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A DISCUSSION OF THE SENSITIVITY OF A METHOD FOR MEASURING IMPEDANCE BY MEANS OF TRANSMISSION LINES: SUGGESTIONS FOR APPLICATION TO THE INVESTIGATION OF LIQUIDS AT ULTRA-HIGH FREQUENCIES

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ABSTRACT. Arrangements are discussed for improving the sensitivity and widening the range of applicability of an existing method of measuring impedances at the ultra-high frequencies. Two convenient alternative methods based on this discussion, and especially suited to observations on liquids, are described.

§ 1. INTRODUCTION

THE method of analysing impedances at the ultra-high frequencies which was recently described by Flint and Williams (1941) was intended for investigations into the properties of liquids at these frequencies. The impedance of a condenser-cell was to be measured both when air-spaced and when liquid-filled. The method gives good results for most impedances, but in its original form it has been found to be rather insensitive for the determination of impedances with large phase angles. The reasons for this, and practical alternative methods, will be given.

§ 2. RÉSUMÉ OF THE ORIGINAL METHOD AND REASONS FOR ITS INSENSITIVITY

The unknown impedance, Z_2, is placed at the end of a transmission line of characteristic impedance Z_0. The current reflection coefficient may be written as

$$\Gamma = \frac{Z_0 - Z_2}{Z_0 + Z_2} = e^{-2(a+jb)},$$

whence

$$Z_2 = Z_0 \tanh(a+jb). \qquad \ldots\ldots(1)$$

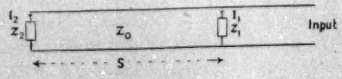

Figure 1.

The constants a, b are determined from the standing-wave system in the following way.

A movable detector (Z_1) is placed across the lines at a distance S from Z_2. I_1 and I_2 are the currents in Z_1 and Z_2 respectively. Put

$$I_1^2/I_2^2=\rho^2 \quad \text{and} \quad 2\pi/\lambda=\beta.$$

It was shown (*loc. cit.* equation (11)) that

$$1-\frac{\rho^2}{\rho^2_{\max}}=\frac{\cos^2(\beta S+b)}{\cosh^2 a},$$

which we may write more conveniently as

$$\rho^2=\frac{A}{2}\,[\cosh 2a-\cos 2(\beta S+b)], \qquad \ldots\ldots(2)$$

where

$$A=\left|\frac{Z_0}{Z_1}\cdot\frac{1}{\cosh(a+jb)}\right|^2. \qquad \ldots\ldots(3)$$

A typical curve of ρ^2 against S is given in figure 2.

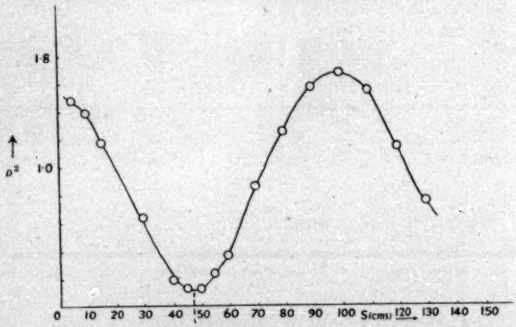

Figure 2.

b is deduced from $S_{\min}$, for

$$b=-2\pi/\lambda\,.\,S_{\min},$$

and may usually be determined with considerable precision (within 0°·5), since in practice the curves are very symmetrical.

With this value of b, a graph of ρ^2 against $\cos 2(\beta S+b)$ is constructed (figure 3). The intercept OB is $\cosh 2a$, and this is certainly the easiest way of deriving the value of a from the data.

Putting $Z_2=R_2+jX_2$, equation (1) gives

$$\left.\begin{aligned} R_2&=Z_0\frac{\sinh 2a}{\cosh 2a+\cos 2b},\\ X_2&=Z_0\frac{\sin 2b}{\cosh 2a+\cos 2b}, \end{aligned}\right\} \qquad \ldots\ldots(4)$$

provided Z_0 is purely resistive. In practice an air-spaced line is used, for which Z_0 is calculated.

The accuracy of determination of b is independent of Z_2. This is not so for a, for which the accuracy depends on that of the extrapolation to B (see figure 3). For large values of a the gradient of the line furnishes a sufficiently

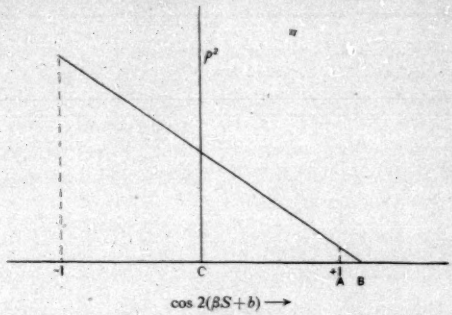

Figure 3.

reliable estimate, but such large values of a are rare in practice, only occurring when the line is almost perfectly matched. Small values of a are much more frequent, as will be seen later. The intercept $AB = 2\sinh^2 a$, so that for small values of a large percentage errors in the value of R_2 can be introduced by the extrapolation. As a guide, $\cosh 2a \geqslant 1{\cdot}5$ is recommended for accurate work.

§3. SUGGESTED WAYS OF IMPROVING THE SENSITIVITY

In an attempt to improve the accuracy it will be advantageous to examine the graphical solution (figure 4) of the basic equation

$$Z_2/Z_0 = \tanh(a+jb) = R+jX.$$

(The equations to the systems of circles are:

$$X^2 + R^2 - 2R\coth 2a + 1 = 0,$$
$$X^2 + R^2 + 2X\coth 2b - 1 = 0.)$$

If we write Z_2 as $|Z_2| \angle\theta$ and Z_0 is purely resistive, then the vector OP in this diagram is $|Z_2/Z_0| \angle\theta$, i.e. it has the same phase angle as the impedance Z_2. This we cannot readily alter, so that the only quantity which could be adjusted is Z_0. It can easily be shown from the equations to the systems of circles quoted above that a has a maximum value when $|Z_0| = |Z_2|$ or $b = 45°$. This condition, which will be referred to as the optimum condition, is independent of θ, and thus applies to all impedances. The graphical significance of this condition is readily seen by reference to figure 4.

The maximum value of $\cosh 2a$ is Z_2/X_2 or $\operatorname{cosec}\theta$, and, therefore, it is essential when making observations on impedances of large phase angles to "match" Z_0 and Z_2 in this way. Fortunately the setting is not critical. This procedure is not so necessary for predominantly resistive impedances.

Since Z_0 cannot conveniently exceed 500 ohms, this constitutes an approximate upper limit to those impedances of large phase angle which can be measured with any accuracy by this method. A wider range, however, is available when θ is small (see figure 4). A suitable type of transmission line would consist of a parallel-wire or parallel-plate system with a variable spacing, but this is scarcely practicable or convenient in use.

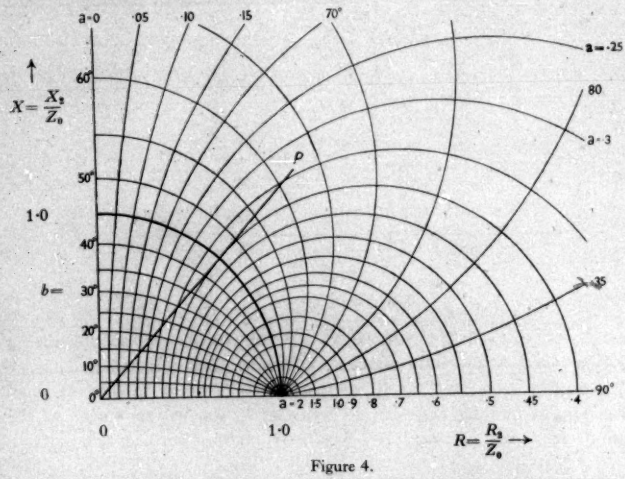

Figure 4.

The scope of the method may, however, be widened by the artificial reduction of the phase angle of the terminating impedances. When Z_2 is predominantly reactive this may be effected by shunting it with a variable length, l, of short-circuited line. Adjustment of l can reduce the phase angle to zero.

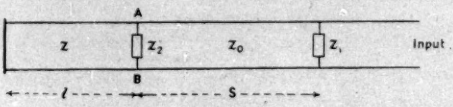

Figure 5.

The input impedance to the extra length, l, is $jZ \tan \beta l$ and the total input impedance, Z_t, at AB is

$$Z_t = \frac{Z \tan \beta l[R_2 Z \tan \beta l + j(R_2^2 + X_2^2 + X_2 Z \tan \beta l)]}{R_2^2 + (X_2 + Z \tan \beta l)^2} \quad \ldots\ldots(5)$$

This is purely resistive when

$$\tan \beta l = -\frac{Z_2^2}{ZX_2} = -\frac{Z_2}{Z} \cdot \operatorname{cosec} \theta, \quad \ldots\ldots(6)$$

and then

$$Z_t = Z_2^2/R_2 = Z_2 \sec\theta, \qquad \text{.....(7)}$$

Also

$$\left.\begin{aligned} \frac{Z_2^2}{R_2 Z_0} &= \tanh a \text{ if } b=0, \text{ i.e. } Z_0 > \frac{Z_2^2}{R_2}, \\ \text{or} \qquad & \coth a \text{ if } b=\pi, \text{ i.e. } Z_0 < \frac{Z_2^2}{R_2}. \end{aligned}\right\} \qquad \text{.....(8)}$$

The value of Z_t at resonance will be high if Z_2 has a large phase angle. In such a case the maximum value of Z_2 for which accurate results could be obtained would be roughly $500\cos\theta$ ohms, so that, as θ increases, the range of impedances which can be dealt with decreases. But within this range it is always possible to make Z_t real, and further, to approximate to the optimum condition. (Of course in this method the case where both Z_0 and Z_t are real and their moduli are equal must be excluded, or our standing-wave system would vanish.) Figure 6 will help to illustrate this point.

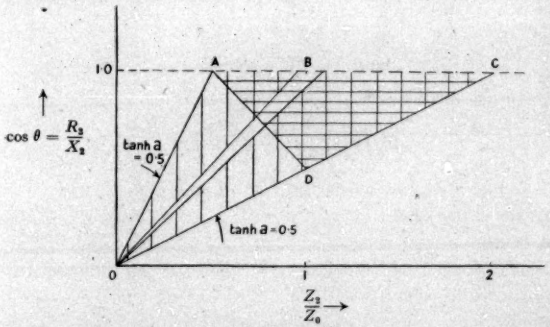

Figure 6.

The slopes of OA, OB, OC are $\tanh a$ when $Z_0 < Z_2^2/R_2$ and $\coth a$ when $Z_0 > Z_2^2/R_2$.

The shaded areas reaching to the origin represent the region of availability of this method based on the practical criteria $1{\cdot}5 \leqslant \cosh 2a \leqslant c.10$. This shows clearly how the range of Z_2 is restricted as θ increases. Using the same criteria, the triangle ACD indicates the availability of the original method, but it should be remembered that the regions shaded should have not sharp but diffuse boundaries.

When using this artifice, the experimental procedure is lengthy and intricate, but the theory of it has been discussed here because it forms the basis of a rapid and convenient method to be described in § 4 (*b*).

Finally, the graphical solution referred to above is easily memorized, and is an exceedingly valuable aid in estimating rapidly the standing-wave system to be expected from any given terminating impedance and *vice versa*. It has also interesting educational potentialities.

§ 4. TWO PRACTICAL METHODS

In view of the difficulties discussed in the previous section, the following practical adaptations of the foregoing theory are being used for the investigations now in progress.

(*a*) *Method* 1. This method is relatively simple in procedure and gives both the dielectric constant and the losses directly. The principal drawback is the quantity of liquid necessary. A concentric line is employed.

Reverting to the basic equation (1),

$$Z_2 = Z_0 \tanh (a+jb),$$

we retain Z_2 as invariant and observe the standing waves when the line is air-spaced and liquid-filled respectively. Then

$$Z_2 = Z_0 \tanh (a+jb) \quad \text{for the air-spaced line,}$$

and

$$Z_2 = Z_0' \tanh (a'+jb') \quad \text{for the liquid-filled line.}$$

(The change in the value of β caused by the introduction of the liquid is allowed for in the experimental determination of a', b'.)

Hence

$$\frac{Z_0}{Z_0'} = \frac{\tanh (a'+jb')}{\tanh (a+jb)} = \frac{R'+jX'}{R+jX}.$$

Also, in the usual notation,

$$\frac{Z_0}{Z_0'} = \sqrt{\left(\frac{L}{C}\right)} \sqrt{\left(\frac{G+jC'\omega}{jL'\omega}\right)},$$

since conductor losses are very small. But $L' = L$ and $C' = \epsilon C$, where ϵ is the dielectric constant of the liquid.

We have, therefore,

$$\sqrt{\left(\frac{G}{jC\omega} + \epsilon\right)} = \frac{R'+jX'}{R+jX} = r'/r \,.\, e^{j(\theta'-\theta)}.$$

(see vector OC in figure 7), i.e.,

$$\left.\begin{aligned} \epsilon &= |r'/r|^2 \cos 2(\theta'-\theta), \\ G/C\omega &= \epsilon \tan\delta = -|r'/r|^2 \sin 2(\theta'-\theta), \end{aligned}\right\} \quad \ldots\ldots (9)$$

where $\tan\delta$ is the loss factor of the line capacity.

In terms of the Cartesian components, these quantities are:

$$\left.\begin{aligned} \epsilon &= \frac{(R^2-X^2)(R'^2-X'^2)+4RXR'X'}{(R^2+X^2)^2}, \\ \epsilon \tan\delta &= \frac{RX(R'^2-X'^2)-R'X'(R^2-X^2)}{(R^2+X^2)^2}. \end{aligned}\right\} \quad \ldots\ldots (10)$$

When the losses are very small, OC will coincide with the real axis, and its length is $\sqrt{\epsilon}$ directly.

This method is simple in technique and has all the advantages of a rapid comparison method, especially when the apparatus has been calibrated, i.e. when the constants a, b for the air-spaced line are known.

For water or other liquids of large dielectric constant, the second vector OB will be very long and lie well off the optimum quadrant. It is recommended that in this case Z_2 should be chosen so as to cause both points A, B (see figure 7) to straddle this quadrant. This will scarcely affect the length of OC, but will ensure a comparable accuracy in the determination of both points A, B.

(*b*) *Method* 2. This method requires less liquid, but is only applicable when the losses are negligible. It is a combination of the previous method with the artifice described in § 3.

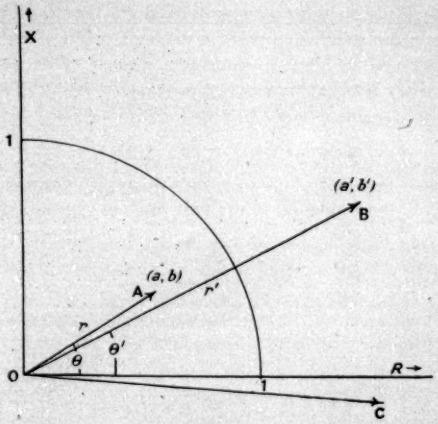

Figure 7.

A concentric line is bridged by a suitable impedance Z_2 and terminated by an adjustable length of "shorted" line which can be filled with the liquid. A crystal probe is placed $\lambda/4$ from Z_2 (λ is determined from an auxiliary wavemeter). The length l of the extra line is adjusted until the crystal records a minimum. This is repeated with the extra line filled with the liquid.

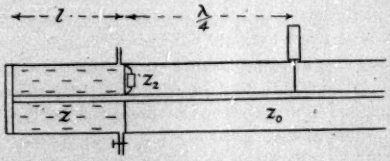

Figure 8.

The minimum at the crystal occurs when Z_t is real and if $Z_t > Z_0$. At resonance, $Z_t = Z_2^2/R^2$, which is invariant.

Also $\tan \beta l = -Z_2^2/ZX_2$ for the air-spaced line, and $\tan \beta' l' = -Z_2^2/Z'X_2$ for the liquid-filled line, provided Z' (the characteristic impedance of the liquid-filled line) is purely real.

Hence $$\frac{Z}{Z'} = \frac{\tan \beta' l'}{\tan \beta l},$$

or $$\sqrt{\epsilon} = \frac{\tan \sqrt{\epsilon} \cdot \beta l'}{\tan \beta l}. \qquad \ldots\ldots (11)$$

This is most easily solved by plotting the simultaneous equations

$$y = \tan \alpha x,$$
$$y = \gamma x,$$

where $x = \sqrt{\epsilon}$, $\alpha = \beta l'$, $\gamma = \tan \beta l$.

Thus $\sqrt{\epsilon}$ depends on the determination of λ, l and l'. The probe is not used as an absolute instrument but as a detector only, so that its ultra-high-frequency calibration need not be known. This is most convenient. This method is singularly rapid, but it is applicable only to liquids with negligible losses. For all others the second method is advocated.

§ 5. ACKNOWLEDGMENT

Finally, I wish to thank Dr. Gordon Williams for numerous helpful discussions, especially those relating to the first method, described in § 4 (*a*).

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SIXTY YEARS OF COLORIMETRY

By G. S. FAWCETT,
The Tintometer, Ltd.

A lecture delivered to the Colour Group 12 *May* 1943

§ 1. INTRODUCTORY

As an instrument manufacturer, I am brought into touch with the problems of many concerned with colour, but I am not going to weary you with a catalogue of these. I propose to begin with some very brief facts about the man without whose work I should not be addressing you to-day, and to whom science and industry are, in no small measure, indebted.

Joseph Williams Lovibond was the inventor of the Tintometer. He was born at Long Sutton, in Somerset, 17 November 1833. When he was five years old his mother died; at 13 he was apprenticed to a Captain Aitken in the Merchant Navy and sailed in an old windjammer. The captain was a pretty tough nut, and after a few voyages Lovibond, always a good swimmer, preferred to dive overboard in an Australian harbour and swim ashore through shark-infested waters rather than endure another voyage with Captain Aitken.

He next went gold prospecting to Ballarat in Australia, and from there worked his passage to America and was gold prospecting in California in 1849. Life there was hard, as can be gathered from the tales which were written about

the camps. In at least one "friendly argument" between different factions he was lucky to get away with his life. On his return journey to England he had the harrowing experience, whilst saying good-bye to his friends ashore, of seeing his savings in gold coin fall from his hat into the sea.

At the age of 21, that is in 1854, he became associated with his father and two brothers in a brewery at Greenwich. This was the start of John Lovibond and Sons, Ltd., and in 1869, Joseph, with his wife and family, moved to Salisbury and bought a brewery there. As a brewer he was naturally anxious to set the highest standard for his products, and to achieve this he realized the necessity of understanding and controlling each process of every operation. He knew, of course, that the flavour of beer is associated with its colour and brightness, the most palatable beers being a bright, clear amber. Naturally the intensity of the colour would differ for different grades or types of beer. This was the source from which Lovibond's interest in colour sprang, and was the inception of the Tintometer.

§ 2. EARLY DEVELOPMENT OF THE LOVIBOND TINTOMETER

The first practical step towards an instrument was a tea caddy—filched, I believe, from my grandmother's kitchen—into which two glasses of beer fitted comfortably. Two rectangular slots cut in back and front of the caddy were all that was necessary for two glasses of beer to be compared critically. This simple device was very useful for comparing one sample with another, but did not provide a standard, for no self-respecting beer will tolerate being repeatedly poured into a glass and returned into a bottle without protesting. It shows its disgust of such unworthy treatment by losing its sparkle or brilliance and becoming flat, cloudy or discoloured. The colour of beer depends principally upon the careful kilning of the barley in the process of converting it to malt.

Malt is tested by making an infusion, known as a malt wort—a very similar process to that of tea-making, a ground sample of the kilned barley taking the place of the tea leaves.

Lovibond, therefore, had two major problems:

1. To devise some means of testing and recording the colour of his malt worts.
2. To provide colour standards for his finished beers.

Solutions of pure chemicals approximating to the colours of the malt worts were prepared, but abandoned on account of instability. Sealed tubes containing solutions were tried, but abandoned for the same reason. Hollow wedge-shaped cells were used to provide increasing intensities of colour, and some very crude forms of solid glass wedges were also tried.

Flat coloured glass gave promise of satisfaction, and after extensive searches through the bins of several glass merchants, Lovibond was successful in finding a few samples of amber-coloured glass which very closely resembled the colour of malt worts and beer. Fortunately, several different intensities of the glass were available, and he found it was possible to match the colour of a dark glass very closely by superimposing two or more lighter glasses. He was so

delighted with the possibilities of this discovery that he ordered large consignments of the glass, and from this constructed the original *Series 52* scale, which is still used in the brewing industry both in this country and in the United States of America.

From the beginning, he realized the necessity of careful control of the conditions of observation, and he constructed an instrument with a viewing tube at one end and two equal rectangular apertures at the other. In front of one of these apertures, in a rectangular vessel, was placed the sample for test, and in front of the other were placed the Lovibond glass colour-slides. The slides were numbered in units from 1 to 20, and consisted of pieces of glass 2″ × ¾″. He then found that this series of amber glasses did not match, to his satisfaction, some of the more heavily kilned malt worts, and he introduced a duller brown series of glasses, which were later termed *Series 50*, but are now obsolete.

Realizing the value of his new instrument, he introduced it to a number of his brewery friends, who, after trying it, reported favourably; suggestions for applying the same principle in other fields were soon forthcoming. The result was that a number of arbitrary glass scales were constructed for testing different substances, and there is little doubt that at that time Lovibond must have been a very good customer of certain glass merchants who would allow him to look over their warehouses. Scales of varying hues were constructed—several different reds, greens, blues, violets, yellows and oranges. But difficulties started to arise; few substances could be tested by any one scale, and it meant combining two or more colours to match many of the samples. This became confusing, and it was soon apparent that there was no end to the number of scales which could be constructed.

It was evidently whilst working with those arbitrarily constructed scales that he realized what a wide range of colours can be produced by combining suitable red, yellow and blue glasses, and he visualized the possibility of constructing one scale consisting of three graduated series of these colours. After consulting authorities on coloured glass, he got in touch with a firm who were prepared to make a number of trial melts, and eventually obtained supplies of flashed red, yellow and blue glasses, with which he found he could produce a satisfactory absorption of light and which, when combined in suitable intensities, appeared grey or neutral.

The absorption of white light formed the basis upon which the scale is constructed. The actual dimension of the unit, i.e. the intensities of red glass, yellow glass and blue glass which absorb one unit of white light, was adopted arbitrarily, but, having been defined by correlating it to certain physical constants, was recoverable.

All the early experimental work was carried out in a summer-house at the rear of his house in St. Ann's Street, Salisbury.

After the brewing industry, the sugar, oil and tanning industries became interested in colour testing, and gradually interest spread further. Lovibond read many papers before scientific societies, and in spite of many disappointments and the expenditure of considerable sums of money, his faith in his invention never failed.

Among his early papers we find "A new method of Colour Analysis",

reported in the *J. Soc. Chem. Ind.* 1890 ; "Colorimetry", read before the British Association at Edinburgh in 1892 ; and many others before the Institute of Brewing, the Society of Dyers and Colorists, the Royal Meteorological Society and the International Society of Leather Trades Chemists.

§3. LATER DEVELOPMENT OF THE TINTOMETER

In his early work he was assisted by his daughters and a small staff of enthusiastic assistants. In 1896 he formed a private limited company under the name of *The Tintometer, Ltd.*

The year before, in 1895, he retired from active management of the Salisbury brewery, remaining Chairman of the company, and moved to Lake House, a fine old Elizabethan manor situated on the Wiltshire Avon, seven miles north of Salisbury. Here he carried on many of his colour investigations, and early in the present century was experimenting in what is now known as camouflage. Long before the last war he was trying to interest the military and naval authorities in concealing their positions by screens painted with daubs of the same constituent colours as the background. The Admiralty were sufficiently interested to afford him the opportunity of carrying out some tests in the Portsmouth area in 1915, and we have in our possession an interesting notebook kept whilst he was on board the Admiralty yacht.

One of my earliest recollections as a small boy is of turning the wheel of a very large colour-top erected in front of a beech wood at the far end of the park facing Lake House. The segments of the colour-top were adjusted until the spinning top blended with the background, and then screens were painted in daubs in proportion to the areas of each colour exposed on the face of the top. Observations were made from my grandfather's bedroom windows.

He was twice Mayor of Salisbury and was J.P. for the city and county. He did much to encourage technical education and home crafts.

A man of extraordinary energy, always up and about before anyone else, interested in whatever was going on, Lovibond kept an open house and delighted to have young people about. Although seriously handicapped by deafness, which increased with age, he kept young in mind and body. He played tennis with his grand-children after he was 70, and rode to a meet of the local foxhounds on his 80th birthday. He died after a short illness in 1917.

During the World War of 1914–18 a number of new tests were developed, and in spite of most of the male staff having joined the forces, production was carried on under the management of Miss Baker, who had worked with Mr. Lovibond for many years. On the death of Mr. Lovibond he was succeeded as chairman by his youngest daughter, now Lady Headley.

During the short industrial boom period following the last war, many new applications of the Tintometer were developed. Thus, in 1927, Drs. Rosenheim and Schuster, of the Medical Research Laboratories, constructed an instrument whereby the glass colour slides can be rapidly changed by mechanical means, for the colorimetric assay of Vitamin A in cod-liver oil and similar substances by the antimony trichloride reaction. The colour is extremely fugitive, reaching a maximum intensity within about 30 seconds of contact with the reagent, and immediately starting to fade. This instrument gave way to the

Lovibond Tintometer, British Drug Houses pattern, which was developed from a model made in the analytical laboratories of the British Drug Houses, Ltd., by Mr. T. Tusting Cocking, who was also working on Vitamin A estimations by the Carr-Price test, which originated in the B.D.H. laboratories.

This test undoubtedly did much to stimulate interest in colorimetric determinations, and created world-wide interest in the Tintometer. It was a most useful outlet for our activities during the slump period which followed the industrial boom.

In 1927 the Bureau of Standards published a spectrophotometric analysis of the Lovibond glasses which has proved a very useful contribution to our knowledge of the properties, defects and requirements of our own slides. Although in many instances critical of the Lovibond glasses, when read carefully it pays high tribute to Lovibond's work.* On page 10 Priest says: "This seems an opportune time for the writer of this preface to state his opinion that lack of uniformity in the glasses now in use is not the major cause of discrepancies said to occur between different laboratories in the colour grading of oil. The need of standardization of the glasses has been greatly emphasized (perhaps relatively over-emphasized) by the chemists concerned in this problem. The uniformity of the glasses used by different chemists in different places is, however, but one of the conditions to be satisfied in order to insure agreement of results in colour grading of oil. It seems worth while to point out the following possible sources of discordant results:—

1. Use of glasses without rule as to the number of separate glasses used to match the oil.
2. Use of colour comparators (tintometers) of widely different types.
3. Use of illuminants of widely different quality.
4. Colour-grading of oils by many persons whose colour sense and power of colour discrimination have never been tested."

It was very largely as a result of the Bureau's publications and of my discussions with Priest and Gibson that we embarked upon the manufacture of the Lovibond Colour Standards in pot metal, i.e. standards with the colour throughout the glass, which enables the glass to be optically worked.

§4. THE C.I.E. (COMMISSION INTERNATIONALE D'ECLAIRAGE) SCALE

The year 1931 was probably one of the dates which will for all time stand out in the history of colorimetry. It was the year the C.I.E. published, among other matters, their recommendations defining a trichromatic system for the expression of colour measurement. Unfortunately this system has not met with the interest and application it deserves, and I believe this is due to the lack

* Unfortunately the purpose of this survey was misunderstood and resented until we met Priest when he visited this country for the C.I.E. Commission in 1931, and I returned with him to the Bureau of Standards late in that year. I am, therefore, glad to acknowledge the benefit we have derived from this publication and the help which I personally have received on two visits to the Bureau of Standards, first during the lifetime of the late Irwin G. Priest and, more recently, from his successor in office, Kasson S. Gibson.

of simple explanation. I should like, therefore, to focus attention on the necessity for a much wider understanding of the principles of the C.I.E. system of colour measurement.

If we want to rouse interest in something complicated, for example in the internal combustion engine, we do not start off with a whole book full of mechanical data, nor go into the strength of materials and the flash-points of various spirits, but we take a great deal for granted and start by explaining general principles—for example, that petrol is fed into a carburettor, where it is mixed with air, which, when ignited in the cylinder, explodes, the force thus exerted acting upon the piston, which sets the rest of the machine in motion. Broad principles such as these are sufficient for some people. The majority will seek after further knowledge, but we must expect the knowledge to be assimilated in relatively small doses. Even an understanding of such broad principles as I have outlined serve a definite purpose, and is the essential first step.

So, in explaining colour measurement, we note first that whatever method is attempted to express colours, it is essential to employ three variables, and it is difficult to think in three dimensions. There are two generally recognized methods of matching colours:

1. Subtractive systems, where one starts with a beam of white light and produces any required colour by introducing into the beam transparencies which selectively absorb that beam. The Tintometer is a subtractive colorimeter.
2. Additive systems, whereby specially selected light sources are focussed on to a non-selective reflecting screen and are mixed in suitable proportions to obtain the required colour. Typical of such systems are the Guild Colorimeter. the Donaldson Colorimeter and the Trichromatic Colorimeter described by Wright.

But, as I said earlier, both types employ three variables.

§ 5. THE COLOUR CUBE

The object of the C.I.E. is to provide a common language of colorimetry by allocating a value to every possible colour the normal eye can appreciate, so that, whatever system is employed, the final answer is expressed on a common basis. It is done in this way: every colour is allocated to a position within a solid figure, and its position defines the colour. For convenience it was recommended to take a cube. In figure 1, O is the origin, X and Z the two horizontal axes, and Y the vertical one. Then any point lying within the cube can definitely be fixed relative to these three lines by measuring the perpendicular distance from each of these axes. This is what, in effect, the C.I.E. system does.

Within the bounds of a cube every colour is allocated to one position. No colour can occupy more than one position. The positions are allocated in this way. In figure 1, because it is a cube, these axes are all of the same length. If these points are joined X to Y, Y to Z and Z to X, and we cut off the corner of the cube so formed, we have exposed an equilateral triangle, X, Y, Z. The properties of any equilateral triangle are such that if the length of the side is taken to represent unity, then the position of any point which lies within that

triangle relative to X, Y and Z, when measured as the perpendicular from the side opposite these points, always totals unity. This means that it is only necessary to know any two values to fix a point; the sum of these two deducted from unity gives the third value.

In the C.I.E. system, each of these angles is made to represent a colour stimulus:

Red at X.
Blue-violet at Y.
Green at Z.

These are theoretical stimuli, of which the values have been calculated. They have been selected so that equal intensities of these three stimuli form white light. The centre of the triangle is, therefore, allocated to white, and that fixes the first point. That is known as the *equi-energy white*. The next step is to plot on this triangle, in terms of these three stimuli, the positions of spectral

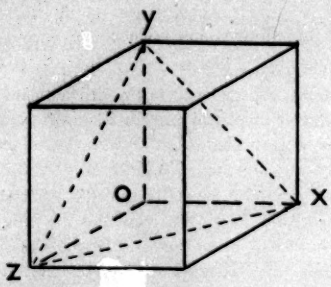

Figure 1.

colours of known wave-lengths. Join these points together to form the *spectral locus*.

Colours which are less saturated than spectral colours will lie somewhere between the spectral locus and the equi-energy white at the centre of the triangle. The position of any point on this figure defines the quality of the colour. The method of stating its position is relative to X, Y and Z. A line drawn from equi-energy white through any point and extended to the spectral locus determines the dominant-hue wave-length. The distance along the line towards the spectral locus determines the saturation of the colour. Figure 2 shows the colour-quality triangle plotted on rectilinear co-ordinates. The outside line is the spectral locus, whilst inside the network is the area which can be covered by Tintometer glasses.

Now the colour quality does not fully define a colour. It also has a brightness value. The best way to understand the relationship of brightness and colour quality is to take any coloured opaque object, illuminate it with a single source of illumination, and then move it nearer to the source and see how the colour brightens, then move away and see how the colour gradually becomes duller. The colour quality has not changed provided the quality of the light has remained

constant. The same effect is produced by casting a shadow on the object. As the light falling upon the object is decreased, the brightness falls, but the colour quality does not change. Brightness is represented in our cube by the perpendicular distance from the origin O. The bottom plane of the figure represents zero brightness.

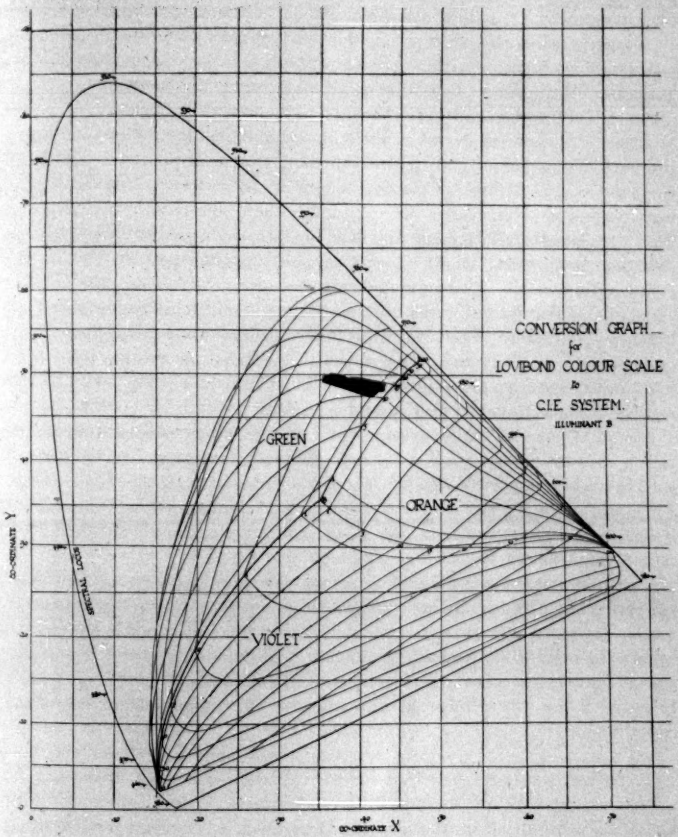

Figure 2.

Now consider the axis of the cube passing through O. At the bottom it is 0% brightness—black. At the top it is 100% brightness—white. We also know it passes through equi-energy white on the colour triangle, so this axis represents the achromatic series, or white with increasing shadows to black. Similarly, any other shadow series for any given colour quality can be

represented by a line drawn from the origin through the colour-quality position on the colour-quality triangle.

Sections through the cube, parallel to the plane of the colour-quality triangle, will show the same relative positions of all the colour qualities, but at different brightness levels.

In the preliminary stage this, I think, is as much as should be attempted in explaining the C.I.E. system to persons without technical training.*

I suggest we should aim at getting this much understood and accepted before attempting to proceed further. In many cases it will be enough. Possibly suggestions for greater simplification or more instructive methods of demonstrating the fundamental principles will be forthcoming.

Can some ingenious person devise a jig-saw puzzle of the colour-solid fitting inside a cube, which demonstrates the fundamental principles as it is built up from the base? I think it should be done in transparent plastics. Having been trained as a surveyor, I like to visualize the colour-quality triangle as an island, and the three imaginary stimuli as lighthouses, focussing their beams on different parts of the island. Then, as one digs downwards towards the centre of the earth, one gets muddier and muddier, and so do the colours. Similarly, as one flies upwards towards the sun, the colours get brighter and brighter. Of course, the analogy breaks down because the two imaginary lighthouses x and z have no brightness value; furthermore, the sun has to be fixed in an unusual position. But if we can get hold of some idea of this sort, I believe we shall have performed a really useful service.

Some of you may have seen examples of the cartoon films for explaining rather abstruse mathematical problems. Several were shown at a meeting of the Physical Society last November, when Mr. Ashhurst read a paper on *Scientific Films and their use in the teaching of Physics*. At the recent symposium of the British Association, a biological film showing the development of the foetus of a rabbit was exhibited.

Is it too much to hope that perhaps one day we might, as a Group, foster the production of a diagrammatic film showing how the C.I.E. system operates? I believe it could be done, and if well done would be of inestimable educational value. In reply to an enquiry, Gaumont British indicate that a reel to run about 11 minutes in colour would, at the present time, probably cost about £1500. It seems a case where financial help might be forthcoming from one of the educational funds.

§6. RECENT DEVELOPMENTS

About a year before the war we were fortunate in getting in touch with Dr. R. K. Schofield, of the Physics Department at Rothamsted. Certain suggestions which he made to us for the modification of our apparatus have enabled us to bring the Lovibond colour scale within the orbit of the C.I.E.

* By suggesting such a course I do not underrate the true significance of all the experimental data compiled by the original workers, the sound reasoning upon which the system has been constructed and, perhaps most of all, the patience and toleration displayed by those concerned, who finally reached agreement or compromise where perfect agreement is not possible, so that the system could be adopted internationally. Our Chairman, Mr. Guild, was undoubtedly the moving spirit, at any rate in this country.

system, so that it can continue to serve science and industry to even greater advantage than was possible in the past.

Schofield had introduced into his Tintometer at Rothamsted an ingenious device, constructed mostly with Meccano, which enabled relative brightness of the two halves of the field of view to be varied at will. By so doing, he found that he could match practically all colours without ever using more than two of the three Lovibond scales available, adjusting the brightness by moving the obturator vane which controlled the relative brightness of the fields of view.

Schofield pointed out that if we made use of one of the standard illuminants of the C.I.E. system, and introduced a method of controlling and measuring the relative intensities of the light on the two halves of the field of view, it would be possible to convert the instrumental readings into C.I.E. values, since, from the data published by the Bureau of Standards, it is possible to compute the x, y, z values not only of the single Lovibond slides, but of all possible combinations of any two.

Mr. J. Hewitt, a research student at the Rothamsted Experimental Station, was recommended as being capable of carrying out the necessary calculations, and for some time he joined our firm and was responsible, under Dr. Schofield's supervision, for drawing these graphs.

Figure 3 is the same triangle as figure 2, projected on to the equal chromaticity plane suggested by D. B. Judd.* It shows the position of all possible combinations of any two Lovibond slides on the colour-quality triangle. It has the advantage that the colours are more evenly distributed, so that whatever part of the triangle one considers, a movement of a given distance represents a like number of least perceptible differences.

In order to convert readings made on the Lovibond-Schofield apparatus into terms of the C.I.E., the procedure is: first of all on this graph ascertain the point at which the Lovibond values meet, the red values being plotted along the line from centre to R, the blue values along the line from centre to B, and the yellow values along from centre to Y. Find the point of intersection where the two colours meet, then take the two cursors (which consist of pieces of fine gut fixed at centres x and z respectively), cause them to intersect at this point, and read off the x and z values from the scales. Similarly, if C.I.E. are quoted, the cursors should be placed on the scales and the Lovibond values read off at the point of intersection.

The graph illustrated is plotted for use with C.I.E. Illuminant B; a further graph is available with Illuminant C as centre.

In order to show how colour measurements can be made on two systems and then compared by conversion into C.I.E. units, we have prepared a graph showing some readings made on paint samples which were originally tested by the Guild colorimeter.† These readings were converted into C.I.E. units, and subsequently the samples were tested in the Lovibond-Schofield apparatus and these readings also converted into C.I.E. units.

Very good agreement is shown in all the "clean" colours, but agreement falls off considerably as one approaches the threshold of black; it is to be noted

* D. B. Judd, *Bureau of Standards of Research*, 14, 41 (1935).

† The graph was exhibited at the lecture, but is unsuitable for reproduction.

that the Guild colorimeter readings were made in 1930 and the others, on samples published by the B.S.I. in 1941, were not made until 1942. Further investigation along these lines will be carried out as opportunity occurs.

§ 7. PRESENT AND FUTURE PROBLEMS

It causes us no small measure of satisfaction to find that the instruments which we produce are able to serve the country at war. The exigencies of war have actually stimulated the application of colorimetric determinations in many fields. Some of these applications are of necessity of a confidential nature, but others are of a routine nature, and among these I can mention the testing of the

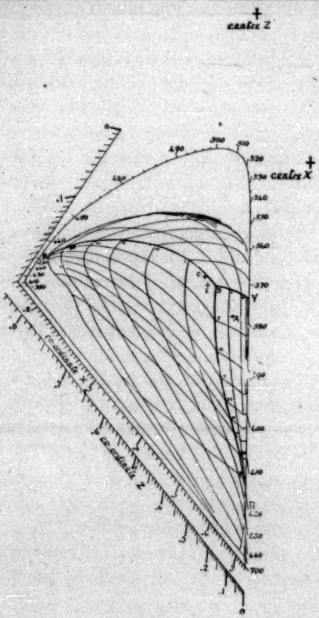

Figure 3.

(Reproduced by courtesy of the 'Journal of Scientific Instruments'.)

chlorination of water, the detection of noxious gases in industry, and the acid wash test on low-boiling fractions of coal tar. Again, for clinical determinations in field hospitals our Lovibond Comparator has proved extremely useful in determining the amount of sulphonamides and other derivatives in body fluids. The apparatus is regularly used to prevent over-dosage by determining the amount of the drug in the blood stream. Yet again, colorimetric tests are used in the food industry, and especially in the testing of pasteurized milk by the phosphatase test, and more recently in a test for determining the keeping quality of milk, known as the Resazurin test.

One of the points to which I should like to draw your particular attention is that all this work has been developed, and our company has been built up, as a result of such a relatively unimportant matter as the determination of the colour of beer.

Owing to the critical conditions of some of the tests, we have always been interested in making rectangular vessels with plane parallel faces. At the time of the introduction of the Vitamin A test, to which I have already referred, the only vessels which would stand up to the antimony trichloride and chloroform were produced in Germany. A colleague of mine, Mr. D. Lovell, not only improved upon the method of production of these sintered glass vessels, but has been successful in developing a system of fusing glassware.

The applications in which fused glassware can be used are almost unlimited, and soon after the outbreak of war a serious position arose because colorimeter cups, which had been of either German or American manufacture, were unobtainable in this country. The Department of Overseas Trade knew that we were producing fused glassware, and asked us to make colorimeter cups, which we did. They then asked us whether we would produce a plunger-type colorimeter. That we have also done. The plunger-type colorimeter is, of course, strictly only a comparator for comparing two solutions, and my only reason for mentioning it here is to draw attention to the wide application of colorimetric tests which may not be apparent to everyone in this Group, most of whom are working upon some specific problem.

Finally, it would not be right to conclude an account of the activities of the Tintometer up to the present day without expressing thanks and paying tribute to many workers in various fields who have helped us to develop colorimetric tests and improve the instrument, and, not least among these, the enthusiastic staff in our works at Salisbury.

DISCUSSION

Dr. R. K. SCHOFIELD. As secretary of an international soil science commission, I was faced with the task of correlating soil-colour readings made by different methods, and decided that the only way was to adopt the C.I.E. system as an ultimate basis. The Tintometer is a particularly convenient apparatus for matching the colour of soil samples, but it was necessary to relate the instrumental readings to the C.I.E. system. If the match is obtained in the original way by combining glasses selected from all three series it is impracticable to provide a conversion graph or table. The solution of the problem lay in adjusting the brightness of the illumination falling on the sample relative to that falling on the white surface, and interposing, between the white surface and the eye, glasses drawn from only two of the series at one time. This restriction greatly reduced the number of combinations of glasses for which the C.I.E. values had to be computed. It also disentangled chromaticity from brightness and resolved the conversion into a two-dimensional graph and a one-dimensional graph. A single three-dimensional graph would be needed on the old system, which could only have been set out as a large number of two-dimensional cross-sections.

According to Judd, the lay-out used in the chromaticity conversion graph is approximately uniform as regards the relationship of perceptible difference to distance, but it retains the C.I.E. coefficients x, y and z as the means of specification. This is achieved by using thread cursors by which the C.I.E. coefficients can be read with greater ease and certainty than if they were represented by a squared background. The graph is, in effect, laid out on a skew perspective of the C.I.E. triangle so that the lines for which x is constant, which are parallel to the side YZ in the C.I.E. triangle, all converge in

perspective to a point on the " horizon ". Similarly for lines of constant y and lines of constant z, except that the former converge on a point so far away that, in practice, only x and z can be found with cursor threads. This is of no consequence, since $y=1-x-z$.

The Tintometer glasses coupled with the conversion graphs provided a unique set of standards which, once made, do not deteriorate unless they are scratched. An extensive range of colours can now be built up to C.I.E. specification. The conversion graphs were computed from three spectrophotometric curves which are now the ultimate standards to which the glasses approximate very closely. From the spectrophotometric curves the spectral distribution of any combination can be obtained should this be required.

Dr. W. D. Wright. I think Mr. Fawcett has shown very clearly that the Tintometer has played, and will continue to play, an important part in scientific and technical measurements of colour. A subtractive colorimeter is, I imagine, likely to have one important advantage over its additive counterpart, namely, that the energy distributions in the two halves of the matching field should in most cases be more nearly alike in the former than in the latter, and hence the measurements should be less affected by abnormalities in the vision of the observer. I should be interested to hear whether Mr. Fawcett has found this to be the case.

I think it is unfortunate that the Tintometer glasses are described as red, yellow and blue, since the true subtractive primaries are magenta, yellow and cyan. It is true that the same terms are used in industry, e.g. in the dyeing industry, but I notice that in the United States the correct terms are now being used more frequently, and I believe it would help in clarifying the difference between additive and subtractive mixture if we could also encourage their wider adoption.

Mr. Fawcett's suggestion of a film is an excellent one, and I hope the Committee of the Colour Group will consider the matter further.

I am not quite clear whether Mr. Fawcett's description of a colour solid is intended to refer to all colours or only to surface colours. While a solid is of great value as a means of relating surface colours, I am doubtful whether it has the same merits for light sources and transparencies. With surface colours, the appearance is relatively constant with change in illumination, state of adaptation and so on, but with light sources this does not hold. 0·0 and 100 per cent brightness are limits which can have no precise meaning unless the state of adaptation is defined, and even then would have little bearing on practical problems of photometry and colorimetry.

I imagine one of the greatest handicaps in the Tintometer system is that the colour of the matching field has to be varied in steps and not continuously. Are the steps sufficiently small for the gaps to be of little consequence in the type of measurement for which the instrument is usually employed ?

Mr. J. G. Holmes. The illuminated chart of the colour triangle shown by Mr. Fawcett is extremely interesting, and makes one wonder what colours might be seen if the physicist could follow the mathematician into the areas beyond the confines of the spectrum locus. A demonstration of this type illustrates the distribution of colour within the spectrum locus in an admirable way and enables the red, yellow, white, green, blue and purple areas to be very readily mapped out. The colours are of course absolute, being derived from the energy distribution of the light transmitted by each filter glass, and it would be interesting to know how the subjective effects are dependent on the state of adaptation of the observer's eyes—such as being accustomed to daylight, equi-energy white, incandescent-filament light, 2360° K., and so on—as the subjective idea of " white " or any other colour is very dependent on the surrounding illumination.

The diagram of the spectrophotometric transmission curves of the standard Lovibond filters showed that the colours are actually magenta, yellow and turquoise-blue, as suggested by Dr. Wright. There are now glass and gelatine filters available, such as those used in additive colorimetry, which are a pure red, a moderately pure green and an indigo-blue, and which select the three wave-length bands from the spectrum with much sharper limits than the filters used in the Lovibond colorimeter. For example, the " red " and the " blue " Lovibond filters both transmit the red and blue ends of the spectrum in different proportions, whereas we can now use a red filter which transmits no blue light and a blue filter which transmits only a very small amount of red light. I doubt, however,

whether such filters would give as wide a flexibility in the use of the instrument as is given by the filters originally chosen by Mr. Lovibond. At that time, the production of coloured glasses had only just reached the stage at which the colours could be regarded as controlled, and the colouring oxides gave absorption spectra similar to those commonly occurring in nature and the natural processes. With these glasses, Mr. Lovibond was able to produce a subtractive colorimeter, working on the same basis of selective absorption as the materials which he wished to measure, and giving good results more easily than would have been possible with the more brilliant colour filters obtainable nowadays.

Mr. H. V. WALTERS. I was interested to hear Mr. Holmes say that blue-green glasses with considerably improved (i.e. lessened) far-red transmission were now available. It seems likely that the sensitivity of the Lovibond instrument in the red and reddish-purple region of the colour field might be somewhat improved if such new glasses were employed. The disadvantage of changing the standards in a commercial instrument are manifest, but I would like to ask Mr. Fawcett whether the Tintometer, Ltd., might consider such a change if a considerable advantage could be shown to accrue from it?

A YELLOW GLASS FILTER FOR COLOUR-TEMPERATURE MEASUREMENTS

By H. G. W. HARDING,
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MS. received 22 *June* 1943

ABSTRACT. A glass filter is described which enables the colour temperature of lamps to be measured, using standard lamps which operate at lower colour-temperatures. The filter is yellow in colour and made from Corning 346 and 507 glasses. The method of construction is described.

Colour-temperature measurements made with the glass filter agree to 4° K. with those using the blue liquid filter previously used at the laboratory. The advantages of the yellow filter over the blue filter are that the yellow filter is more stable, the colour-matching properties are better, the energy match is better for a colour match, the temperature coefficient is a fifth of that of the blue filter, and the yellow filter is placed in front of the brighter lamp, so that, for a colour match, the lamps are nearly equidistant from the photometer.

The properties of glass filters made from (*a*) a yellow glass with Zeiss U.G.3, (*b*) Corning 346 with Chance O.V.2, and (*c*) Cobalt blue glass are mentioned.

§ 1. INTRODUCTION

THE glass filter described in this paper has been designed and made to facilitate the calibration of tungsten-filament lamps for colour temperature. As both the properties and function of the filter are closely linked with the method of realizing the colour-temperature scale adopted at this laboratory, a brief outline of the method will be given. Standard lamps are first calibrated against a full-radiator furnace. The calibration consists of varying the voltage on the lamp until a colour match is obtained between the light from the furnace and the light from the lamp. Sufficiently high precision in the matching of the colours can be obtained visually by the employment of the Lummer-Brodhun photometer. At the colour-matching voltage, the lamp

is said to be emitting light of colour temperature of the same numerical value as the temperature of the furnace. By varying the temperature of the furnace, the relation between voltage and colour temperature is obtained for the standards. These standard lamps, together with their calibration, constitute the colour standards of the laboratory. Other lamps are calibrated by matching them with the standard lamps.

The portion of the colour-temperature scale which is covered by tungsten-filament lamps ranges from about 1800° K. to 3200° K. This range is divisible naturally into two portions, 1800° K. to 2400° K. and 2400° K. to 3200° K. The lower range is associated with the now rather old-fashioned vacuum tungsten lamps, while the higher can only be reached by lamps of the gas-filled type. Vacuum lamps make very satisfactory standards for the colour-temperature range of 1800° K. to 2400° K. If they are treated with care their properties remain constant over very long periods. Gas-filled lamps, however, cannot be considered as satisfactory standards to operate in the region of 3200° K., since their colour characteristics are likely to be unstable. For this reason, coupled with the fact that full-radiator furnaces are more difficult to operate in the higher ranges, a decision was taken at the National Physical Laboratory not to maintain gas-filled standards, but to extend the lower range by what may be called a filter method.

In the range of temperatures considered here it follows from the radiation laws that if a filter has such spectral transmission properties that it transforms exactly one Planckian distribution corresponding to a temperature T_1 into another T_2, then it will transform very closely a third T_3 into a fourth T_4, where

$$\frac{1}{T_1} - \frac{1}{T_2} = \frac{1}{T_3} - \frac{1}{T_4}.$$

This property, that the filtered light is always Planckian, makes possible the use of a single filter with a lamp of varying colour-temperature. A filter which has a step approximately from the upper to the lower useful limits of the range of the vacuum standards, when used in conjunction with these standards, almost completely covers the gas-filled lamp range. The upper limits of the gas-filled lamp range can be reached and exceeded if the filter is used with gas-filled lamps operating at low colour-temperatures, where the colour characteristics can be considered stable.

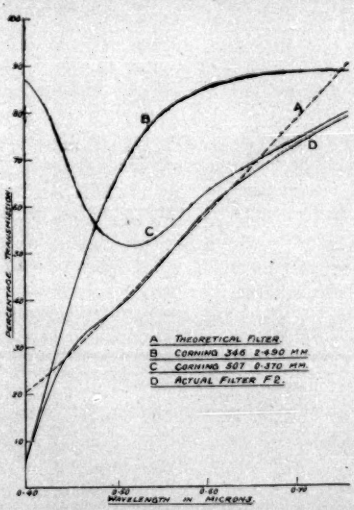
Liquid filters (Guild, 1930–31), that is, coloured solutions of defined strength in parallel-sided glass cells (Donaldson, 1933), have been successfully produced and used for this purpose. The glass filter is intended as a preferable substitute for the liquid filter.

§ 2. CHOICE OF GLASS

The liquid filter transforms the lower colour-temperature of the vacuum standard to the higher colour-temperature of the gas-filled lamp and appears bluish in colour. Accordingly the light observed when colour matching has the colour of the unfiltered light from a gas-filled lamp. It is equally satisfactory, however, to transform the higher colour-temperature into the lower. In this case the filter is yellowish in colour and filters the light from the lamp

at the higher temperature, so that colour matching is made at vacuum-lamp colour. An examination of the available glasses listed in the Chance, Corning and Zeiss catalogues showed that there were no satisfactory blue glasses; attention was accordingly turned to the alternative of a yellow filter.

The ideal or theoretical spectral transmission curve for the filter is given by the quotient of the Planckian energy distributions corresponding to the lower and higher temperatures. The step of the glass filter described is from 2848° K. to 2280° K., and the theoretical transmission curve associated with it is shown as curve A in the figure. The colour-temperature step was intended to be 2848° K. to 2250° K., which is larger than that of 2339° K. to 2848° K. for the liquid filter (filled in 1936 and measured in 1939), since this larger step enables lamps to be calibrated at 3000° K. with vacuum lamps operating below 2400° K.



To reproduce curve A of the figure throughout the spectrum to the standard of visual photometric discrimination, namely less than 1%, would be almost impossible, and is, in fact, not necessary. The colour of the filter must be correct within the limits of colour discrimination, for if the observer is faced with a residual colour difference when colour matching, a decrease in the accuracy of setting will occur: but, provided there is a colour match, larger differences in energy match than 1% can be tolerated. The effect of large energy differences is that the colour-vision characteristics of the observer have influence on the matching point, and so introduce a personal error into the result. It is difficult to lay down an upper limit to departures from a spectral energy match that can be tolerated. Experience has shown, however, that the errors exhibited by the liquid and glass filters are innocuous, whereas those for cobalt glass are serious.

It is unlikely that a single component filter can be made to "colour match" exactly two lamps operating at different specified temperatures, since a liquid or glass having the required properties probably does not exist. With two components, however, an exact colour match can be obtained provided that there is a control over the properties of each component. With the liquid filter the concentration of the liquids provides a suitable control and with the glasses it is the thicknesses of the glasses that are varied. One component of the glass filter described is a yellow-coloured glass, Corning 346, and its transmission curve is shown in the figure, curve B. In colour it is too green, so that a purple glass is required as a corrective. The absorption curve of the purple, Corning 507, is also shown in the figure, curve C. Once the two components have been chosen, the thicknesses are defined by the colour requirements, and there is no further control over the energy match. The art of constructing a successful filter consists in choosing out of the known yellow and purple glasses two that give a reasonable energy match as well as a colour match.

A filter which is designed to give a colour match for a particular part of the colour-temperature scale may not give a colour match at another. For example, a lamp at 3200° K. with a yellow filter in front of it may colour match another lamp at 2500° K., but if the colour temperature of the lamp at 3200° K. is lowered to 2360° K. no colour match may be possible for any temperature of the matching lamp. The colour difference between the matching fields of the photometer may be estimated to be a minimum when the lamp previously at 2500° K. is at 1960° K., and variations of temperature about this value may make one matching field first redder then bluer than the other, but there may always be a residual greenness. This colour-matching property, which is dependent on the original energy match, serves as a guide to the selection of suitable glasses.

There is an additional guide which comes from the colour-temperature step of the filter, as follows. A formula often used to interpolate the colour-temperature step of a filter between the known values for a few illuminants is

$$\text{Reciprocal filter step} = \frac{1}{\theta_C} - \frac{1}{\theta_I},$$

where θ_I and θ_C are the colour temperatures corresponding to the initial and modified colour of the illuminant. The theoretical filter, that is the one which converts the energy from the illuminant to that of the colour-matching illuminant, gives the reciprocal filter step a constant value, but for actual filters the value varies, and the magnitude of the variation depends on the types of glass used.

Where there is little to choose between combinations of glasses as regards the first requirements, this last consideration (that the magnitude of the reciprocal filter step be independent of the illuminant) can serve as a useful guide in the selection of the final glasses.

The two Corning glasses, 346 and 507, are only one pair out of a number that have been investigated, but they are considered the most satisfactory pair from the point of view of these three conditions. Among the other filters considered were Zeiss U.G. 3 together with a yellow which could not be obtained in optical quality glass. This filter, made in 1939, is subsequently referred to

as filter F.1. A combination of Corning 346 and Chance Light Purple (O.V.2) was also considered. Estey (1936) describes other filters which are designed for colour-temperature measurements.

§ 3. PREPARATION OF THE FILTER F.2 FROM CORNING 346 AND 507 GLASS

Two filters have so far been made from this glass, of which one is in use at the National Physical Laboratory and has given very satisfactory results.

The colour-temperature step of the filter was chosen to make a lamp operating at 2848° K. colour match one at about 2250° K.

In the preliminary preparation the relative energy distributions for the 2848° K. and 2250° K. illuminants were evaluated from Planck's formula,

$$E_\lambda d\lambda = C_1\lambda^{-5}[e^{C_2/\lambda\theta} - 1]^{-1}.d\lambda,$$

and the colour of the 2250° K. illuminant calculated using the "Condensed Tables for Colour Computation" (Smith, 1934). The colours of Planckian radiators were also calculated to six figures at 10° K. intervals over the temperature range 1500° K. to 9000° K. This range is in excess of that required for this investigation, but was useful for other work in connection with the colours of illuminants. The value given to C_2 in Planck's formula was 14350.

Plates of the two glasses were polished, the 346 to 3·0 mm. and the 507 to 1·0 mm. thickness. The spectral transmission for each plate was determined. The reliability of these measurements needs to be better than 1%. A photo-electric spectrophotometer assembled by the author using an electrical circuit somewhat similar to one which has already been described (Preston and Cuckow, 1936), but with an altered optical system, was found to be suitable for these measurements. Temperature control of the order of 1° C. is necessary.

The thickness of each glass required for the composite filter used with a 2848° K. illuminant to yield the colour of a 2250° K. Planckian radiator was then calculated. For this filter, F.2, the thicknesses were 2·490 mm. for the 346 and 0·370 mm. for the 507 glass. Different meltings of the glass will probably give values which vary slightly from these.

The plate of purple glass was very thin and could not easily be cemented to the yellow glass without distortion. A plate of hard crown glass about 2·00 mm. thick was therefore prepared, so that the purple could be cemented between the yellow and clear glasses, enabling good optical definition to be obtained through the filter.

The purple glass was prepared first and its thickness measured. This glass was so thin that it was not easy to re-work it to another thickness; allowance was made for the worked thickness by adding to or subtracting from the thickness of the yellow glass, so that the colour of the composite filter for the illuminant 2848° K. still came on the Planckian locus. Adjustment was necessary, so that the colour of the filter is the same as that of a Planckian radiator a few degrees from 2250° K. To allow for possible non-homogeneities in the glass, the effect of the Canada balsam and of the hard crown glass, the yellow glass was worked 0·2 mm. in excess of the calculated thickness and the filter cemented together. Measurements of the spectral transmission values of the completed filter were

then made and the colour determined for the 2848° K. Planckian illuminant. The thickness of the yellow glass to bring the colour on the Planckian locus was calculated. It was found to be desirable to thin the yellow in two stages with a colour determination at the end of each, since, if the yellow is too thin, the colour becomes too purple and the whole process has to be repeated, using a fresh piece of yellow glass.

The colour of the final filter, for a 2848° K. illuminant, should be within 0·00005 Y of the Planckian locus if different observers with reasonably normal colour-vision are to agree in lamp settings to about 1° K. An explanation of the meaning of this colour difference is given later.

§4. PROPERTIES OF THE FILTER F.2

1. *Spectral transmission* (20° C.)

Table 1

Wave-length in microns	Percentage transmission	Wave-length in microns	Percentage transmission
0·39	1·38	0·58	54·86
		0·59	57·04
0·40	5·82		
0·41	12·53	0·60	59·22
0·42	18·41	0·61	61·14
0·43	23·30	0·62	62·86
0·44	27·09	0·63	64·29
0·45	29·95	0·64	65·60
0·46	32·10	0·65	66·74
0·47	33·90	0·66	68·10
0·48	35·27	0·67	69·47
0·49	36·82	0·68	70·83
		0·69	72·12
0·50	38·31		
0·51	39·99	0·70	73·5
0·52	41·70	0·71	74·4
0·53	43·56	0·72	75·5
0·54	45·77	0·73	76·4
0·55	48·04	0·74	77·3
0·56	50·37	0·75	78·4
0·57	52·69	0·76	79·4
		0·77	80·3

2. *Colour for various illuminants*

The bracketed values are the colour coordinates of the Planckian radiators corresponding to the equivalent colour-temperature given in the last column of table 2.

3. *Temperature coefficient*

The change in the colour-temperature step is seen from table 2 to be about 2·1° K. (estimated at 2280° K.) for 11° C. change in temperature.

Table 2

Temperature of filter (° c.)	Illuminant (° K.)	Colour quality	Percentage transmission	Equivalent colour-temperature (° K.)
20	3200	0·47664 X+0·41366 Y+0·10970 Z (0·47663 X+0·41364 Y+0·10973 Z)	52·2	2497·5
20	2848	0·49708 X+0·41526 Y+0·08766 Z (0·49708 X+0·41525 Y+0·08767 Z)	52·8	2279·7
20	2550	0·51680 X+0·41456 Y+0·06864 Z (0·51679 X+0·41454 Y+0·06867 Z)	53·4	2086·5
20	2360	0·53064 X+0·41269 Y+0·05667 Z (0·53063 X+0·41265 Y+0·05672 Z)	53·9	1958·8
31	2848	0·49729 X+0·41527 Y+0·08744 Z (0·49728 X+0·41526 Y+0·08746 Z)	52·6	2277·6

4. *Stability*

This filter was made in April of 1941, and measurements of the colour over a period of two years show that the colour change is small, as shown in table 3.

Table 3

Date	Illuminant (° K.)	Colour quality	Equivalent colour-temperature (° K.)
April 1941	2848	0·49708 X+0·41526 Y+0·08766 Z (0·49708 X+0·41525 Y+0·08767 Z)	2279·7
May 1943	2848	0·49713 X+0·41529 Y+0·08758 Z (0·49712 X+0·41525 Y+0·08763 Z)	2279·3

5. *Colour-matching properties*

The colour-matching properties of the filter are shown in table 2, where it is seen that whatever the illuminant between 3200° K. and 2360° K. the colour does not depart from the Planckian locus by more than 0·00001, 0·00004 and 0·00005 in the X, Y and Z coefficients respectively. These amounts are not visually significant. A comparison of the properties of this filter, F.2, with others may be interesting. For the purpose of the comparison the departure of the colour of the filter for a given illuminant from that of the Planckian locus is measured in terms of the difference in the Y-coefficients of the colour of the filter and the colour of that Planckian radiator which has the same X coefficient as the colour of the filter. Between 2360° K. and 3200° K. the colour of the filter F.2 does not depart by more than 0·00004 Y from the Planckian locus.

The corresponding departures for the filter F.1, made from U.G. 3 and a yellow, are 0·00033, 0·00008 and 0·00028 for 3200° K., 2848° K. and 2320° K. illuminants respectively. A filter using Chance Light Purple O.V.2 and Corning 346 exhibits almost the same characteristics as F.1, and departures of this order are considered to be about the maximum that can be tolerated.

6. *Energy match*

The energy match when a colour match is made with the filter F.2 can be seen from table 4; other filters are included for comparison purposes, and the results show that it is not desirable to use cobalt glass even if there is a good colour match. For each filter the colour step is about 2848° K. to 2320° K., or for the blue filters from 2320° K. to 2848° K. The values quoted are the spectral transmissions of the actual filters divided by the spectral transmissions of ideal

Table 4

Wave-length in microns	Ratio of the spectral transmissions of actual filters to those of ideal filters				
	F.2 Corning 346 and 507	F.1 Original yellow and Zeiss U.G.3	Corning 346 and Chance O.V.2	Blue liquid filter	Cobalt blue glass
0·40	0·28	0·87	0·53	0·84	0·74
0·42	0·78	0·98	1·00	0·97	0·84
0·44	0·99	0·98	1·05	1·00	0·94
0·46	1·04	1·03	1·02	1·02	1·04
0·48	1·02	1·02	0·96	1·01	1·06
0·50	0·98	0·97	0·94	0·98	1·03
0·52	0·99	0·97	0·97	0·94	0·94
0·54	0·99	0·99	0·99	0·99	0·95
0·56	1·00	1·00	1·01	1·03	1·16
0·58	1·01	1·02	1·02	1·03	0·96
0·60	1·02	1·03	1·01	1·01	0·87
0·62	1·01	1·01	1·00	0·99	0·98
0·64	0·99	0·97	0·97	0·97	1·00
0·66	0·97	0·94	0·96	0·96	1·17
0·68	0·95	0·91	0·96	0·93	1·76
0·70	0·94	0·87	0·95	0·89	2·48
0·72	0·91	0·84	0·94	0·83	2·85
0·74	0·89	0·80	0·92	0·77	3·03
0·76	0·88	0·77	0·90	0·74	3·18

filters having the same total transmissions as the actual filters. The spectral transmission of the ideal filter is obtained by dividing the spectral energy distribution values of the Planckian radiator, which corresponds to the colour of the filter for a given illuminant by those for the illuminant.

Calculation of the colours of the filters for various Planckian illuminants shows that the colour-matching properties of F.1 and the filter made from Corning 346 and Chance O.V.2 are about the same, but are worse than those of F.2.

§ 5. TESTING THE FILTER

The filters were checked in two stages, firstly the blue liquid and the F.1 filters were compared, secondly F.1 was compared with F.2. This was necessary because the blue filter was first replaced by the yellow filter, F.1, made partly from Zeiss glass; then, some months afterwards, F.1 was replaced by the yellow filter, F.2, made from Corning glass.

The first comparison of the liquid and F.1 filters was made immediately after the filter steps and temperature coefficients had been determined from spectral-transmission measurements. The temperature ranges were approximately from 2848° K. to 2320° K. and 2310° K. to 1950° K.

A lamp at 2848° K. in combination with the yellow filter was colour matched with a calibrated lamp. The yellow filter was then removed, the blue filter placed in front of the calibrated lamp, and a colour match again obtained. The colour-temperature change corresponding to the change in voltage of the calibrated lamp was determined. Since the constants of the two filters were known, the expected change of colour temperature was known. Experimental and theoretical results agreed to 1° K. at 2300° K.

Similar measurements were then made with the lower-temperature lamp at 1950° K. The agreement was to 2° K. at 2300° K.

The colour-matching properties of F.1, while better than those for the liquid filter, are not exceptionally good—that is, when the voltage of the vacuum lamp is altered to colour match it with the lamp and filter combination there is a residual colour difference of about 0·0003 Y in the photometer fields at some parts of the colour-temperature scale. This colour difference might cause differences in the settings made by different observers. Plates of almost colourless glass (slightly green) were therefore used to correct the more purple of the illuminants, so that the colour differences did not exceed about 0·0001 Y.

In the second stage, comparing F.1 and F.2, the colour-temperature ranges were approximately from 2848° K. to 2280° K. and 2360° K. to 1960° K. At the high temperatures the filters agreed to 3° K. and at the low temperatures to 1° K. of the calculated values.

Agreement between the results expected and calculated was considered to be satisfactory. Results for the first stage were the mean of those for three observers: R.D., A.H. and H.G.W.H. All the measurements in the second stage were made by H.G.W.H. only. The results quoted are the mean of those using different pairs of lamps from the set of six 60-w. 100-v. vacuum lamps used as standards for the range 1800° K. to 2400° K. The lamps operated at 2848° K. were 100-v. 500-w. projectors.

§ 6. ADVANTAGES OF THE YELLOW GLASS FILTER OVER THE BLUE LIQUID FILTER

(1) The filter is made of glass and the colour is stable. This is important from the maintenance point of view. The colour of the glass filter changes as much in two years as that of the liquid filter changes in a month. Although no signs of evaporation are obvious in a cell filled with liquids in 1936, slight uneven dirtiness of the plates in contact with the liquid develops, and this causes difficulties in the spectrophotometry of the solutions. The rate of ageing of the

filters means that the constants of the yellow filter need only be determined about once every two years against once a month for the liquid filter.

(2) The departure of the colours of all of the yellow filters so far considered from the Planckian locus for various Planckian illuminants is about a half or less of those for the blue filter. This means that the yellow filter generally has better colour-matching properties than the liquid filter.

(3) The energy match for a colour match is better for the yellow than the blue filter, except at the extreme blue end of the spectrum. The bad energy match in this region does not appear to affect the usefulness of the filter.

(4) The temperature coefficient of the yellow filter is about a fifth of that for the liquid filter.

(5) The yellow filter is placed in front of the lamp which operates at the higher colour-temperature. This means that both lamps can be kept at reasonable distances from the photometer screens. The blue filter has to be placed in front of the lamp operating at the lower temperature, so that in many instances it has to be much closer to the photometer screen than is desirable.

ACKNOWLEDGMENTS

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A NEW METHOD OF MEASURING THE INCLINATION OF THE EARTH'S MAGNETIC FIELD

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ABSTRACT. The instrument is based on the increase in impedance, due to the skin effect, caused by a decrease in axial component of magnetic field in a wire of high permeability nickel iron when carrying alternating current.

Four straight wires of similar characteristics are arranged accurately in a square, and form the four arms of an A.C. bridge. If the plane of the assembly lies in the magnetic meridian plane and is then rotated about a horizontal axis it is evident that when the total field vector lies in one of the diagonals of the square the bridge automatically balances and the angle between the diagonal and the horizontal plane is the angle of dip.

Such an " all-magnetic " bridge needs no ancillary balancing adjusters.

In the instrument the four wires are mounted as a square assembly in a disc, which is itself mounted in a ring within which it can rotate about an axis lying in one of the diagonals of the square. The ring is supported in a base block in such a manner that by means of a worm and micrometer head the whole ring (carrying the disc) can be rotated about an axis through its centre, perpendicular to the plane of the ring.

The instrument, owing to its inherent geometrical simplicity, eliminates errors due to :—

(*a*) want of perfect match of the wires ;

(*b*) inaccuracy of geometric mounting of the wires ;

(*c*) the vertical axis not being normal to the base plane.

The value of the dip is independent of the adjustment of the electrical unit provided this does not occur between the times of the direct and reversed readings necessary to a determination of the dip.

§ 1. INTRODUCTION

It has been shown (Harrison *et al.*, 1936) that the skin effect in a wire of high-permeability nickel iron can be responsible for quite a large decrease in the impedance of the wire when the component of magnetic field along its axis increases. In particular, if a suitably heat-treated 26 s.w.g. nickel-iron wire, 9 in. long, of the permalloy range carries alternating current at audio-frequency, the change in impedance may be as large as 18% for a change of field from 0 to 0·2 oersted.

This principle was applied by Harrison and Rowe (1938) and by Turney and Cousins (1938) to the construction of various types of magnetometer for measuring any desired component of the earth's magnetic field or of larger fields up to 10 oersteds or more. A null method is used, and the accuracy attained with the type of instrument referred to is about ½% of the earth's vertical component. In these applications a single straight wire *element* is

employed. It forms one arm of an inductance-resistance A.C. bridge with two equal ratio arms, the fourth arm consisting of adjustable inductance and resistance for balancing purposes. The bridge output is amplified, filtered and rectified, the final indicator being a central-zero micro-ammeter.

§2. THE ALL-MAGNETIC SQUARE BRIDGE

When measuring angular changes in a magnetic vector, such, for example, as the change in dip of the earth's total field between two different places, there are obvious advantages in using two straight magnetic elements inclined to one another at a fixed angle and arranged in adjacent arms of the bridge (Turney, Harrison and Serby, 1936).

This disposition of the wires doubles the sensitivity and renders the bridge more symmetrical, thereby diminishing the effects of changes of input current and of temperature.

An extension of the idea leads to the all-magnetic square bridge, in which four matched nickel-iron wires form the arms, the wires being very carefully arranged as a square assembly * lying in one plane or in two parallel planes. No adjustable resistance or inductance components are present.

If the wires are assumed to be accurately matched as regards (impedance, field) characteristics and absolute impedances in any field, and if the geometry of the assembly is good, then the all-magnetic square-bridge will be expected to enjoy the following advantages:

(i) It will be twice as sensitive to changes in field as its immediate forerunner, the bridge with two magnetic arms.

(ii) It will be compensated for temperature changes.

(iii) It will be nearly independent of frequency changes in the supply current and might be expected to be very insensitive to the presence of harmonics generated (of necessity) by itself.

(iv) It will automatically balance in two azimuth positions 180° apart when the plane of the assembly is perpendicular to the plane of the magnetic meridian; thus the device finds the meridian plane.

(v) It will also balance in two azimuth positions when the plane of the assembly is vertical, provided that the line bisecting the opposite angles of the bridge makes with the horizontal plane an angle greater than the dip. The mean of these two azimuth settings is unique and the instrument will find the magnetic north.

(vi) If the assembly lies in the meridian plane, the bridge will again balance when the total field vector passes through either of the two pairs of opposite corners of the square. Thus the dip is determined.

§3. GENERAL ARRANGEMENT

Four 6-in. 26 s.w.g. wires, chosen to be as well matched † as possible, are

* This square assembly in one plane is obviously a special case of more general arrangements of four wires in space.

† The expression *matched wires* means that their impedance in zero field, their (impedance, current) curves and their (impedance, field) curves between 0 and 0·5 oersted, and at an assigned A.C. frequency, are, as nearly as possible, the same for each wire.

mounted in four grooves machined in a bakelite disc 2 cm. thick. The mounting technique is that previously described (Harrison and Rowe, 1938), and the four copper spirals at one end of each of the four wires are made as nearly equal in impedance as possible; the four arms of the bridge thus formed are connected by very low impedance connectors.

A convenient disposition of the wires is shown in figure 1; they are arranged in two accurately parallel pairs, one pair near to each face of the disc; the two

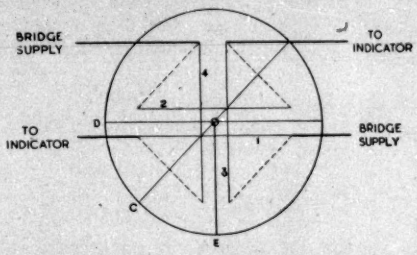

Figure 1. 1, 2, 3 and 4 are NiFe wires embedded in a bakelite disc. The dotted lines are very low impedance connectors.

pairs are accurately at right angles to each other and the two wires of each pair are 6 cm. apart and 3 mm. below the surfaces of the disc. The disc D, figure 2, is mounted in a brass ring R, on a diameter AC as axis, about which it can be rotated inside the ring. The diameter AC lies in one of the diagonals of the

Figure 2.

square bridge and forms an index line. The ring R is mounted on a block B which is fixed to an azimuth circle E. By means of a spring-controlled plunger P on the ring it is possible to rotate the disc about the diameter AC through 180°. The plunger can engage in either of two holes on the circumference of the disc,

of which one is shown at *h*, and locates accurately the two positions. By means of a worm gear and micrometer head, W, the ring itself can be rotated about a horizontal axis passing through its centre and perpendicular to its plane; thus the circumference of the ring carrying with it the axis AC can move relative to B. The lower portion, S, of the ring is graduated in degrees from 90° to 50° on both sides of the 90° graduation, in such manner that when the 90° division coincides with a fiducial mark on the block B the axis AC or index line is normal to the azimuth circle E. Reference to figures 1 and 2 will show how the axis AC bisects the angles of the square bridge; thus, if the bridge balances when this axis lies in the total field vector, the reading on the scale S gives the dip angle direct.

The azimuth circle is provided with two bubbles for levelling; one a 20-second bubble parallel to the plane of the ring R, the other a 1-minute bubble perpendicular to the plane of the ring R.

The electrical unit, which, by means of a valve oscillator, supplies the bridge input current and which amplifies and rectifies the bridge output, is provided with a central-zero microammeter which forms the indicator of the bridge. The unit is connected to the dip meter by a shielded four-core cable whose necessary maximum length is determined by the maximum magnetic field (due to the unit) which can be tolerated at the dip meter. In the instrument described this cable length was 5 metres. The cable is connected to the dip instrument by a short length of light flex which plugs into a socket at O.

§4. CONDITION FOR BALANCE OF THE BRIDGE

A. *Matching of the wires and geometrical construction assumed perfect*

It is now convenient to discuss the conditions under which the bridge can balance, on the assumption that the four wires are perfectly matched and that the construction is geometrically perfect.

Let figure 1 represent the disc, whose axes of rotation, as already described, pass through its centre.

Since it is possible to represent every position of the disc as a great circle lying on a sphere, then, if it be assumed that the magnetic field is uniform throughout the sphere, the problem of finding the field in any wire is one of direction only, and can be conveniently solved by the methods of spherical trigonometry.

Consider, therefore, the wires to be replaced by parallel lines through the centre O of the sphere of diameter ZZ′ (figure 3 *a*).

Let OC be the index line when the bridge is balanced,

ZTZ′ the plane of the magnetic meridian,

ZCZ′ the vertical plane containing the index line of the disc,

P and Q the points at which these planes cut the horizontal,

OT the total field vector dipping $i \equiv \widehat{POT}$,

ϕ the angle between the vertical plane containing the index line of the disc and the plane of the meridian, and

$\theta \equiv \widehat{QOC}$ the reading on the dip-meter scale (S, figure 2) when the bridge balances.

The bridge will obviously balance when the total field vector lies in the plane passing through the index and perpendicular to the disc.

We then have in the spherical triangle ZTC

$$c = 90 + i, \quad t = 90 + \theta \quad \text{and} \quad Z = \phi.$$

Since C is a right angle,

$$\tan t = \cos Z \tan c,$$

or

$$\cot\theta = \cos\phi \cot i. \qquad \ldots\ldots(1)$$

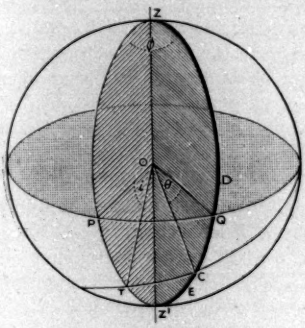

Figure 3 *a*.

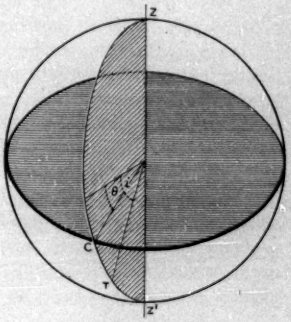

Figure 3 *b*.

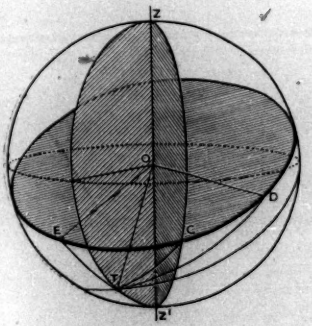

Figure 3 *c*.

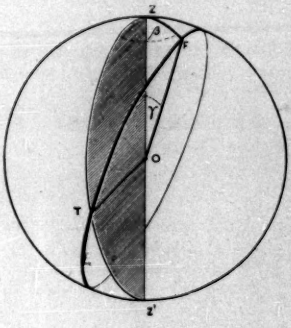

Figure 3 *d*.

Hence we conclude:

Case I. Disc vertical in meridian plane.

If $\phi = 0, \quad \theta = n\pi + i,$

i.e., if the disc is set in the meridian and the base plane is horizontal, i can be read directly on the scale of the instrument.

Case II. Disc vertical, not in meridian plane.

If θ is fixed at any angle $\alpha > i$,

$$\tan(90° + \alpha) = \cos\phi \tan(90° + i),$$

$$\phi = \pm \cos^{-1} \frac{\tan(90° + \alpha)}{\tan(90° + i)},$$

i.e., there will be two balance positions at equal angles to the meridian plane, and the meridian can be found by taking their mean.

Case III. Disc perpendicular to plane of the ring.

In figure 3*b* let the disc be rotated about its index line OC until it is perpendicular to the vertical plane. Remembering that the condition for balance is that the total field should lie in the plane through OC perpendicular to the disc, then the bridge balances when the planes ZCZ′ and ZTZ′ (the meridian plane) coincide.

B. *Matching of the wires imperfect; geometrical construction perfect*

It is not practicable to find four wires sufficiently well matched to fulfil the condition contemplated in case I above; it is therefore necessary to examine the behaviour of the square assembly when the wires do not match. Suppose the assembly is in any position in a uniform magnetic field. Let OP, figure 4 B, be the line in which the total field vector lies, and let OA, the index line, be one of the diagonals of the square.

If OP makes angle ψ with the index, two adjacent wires of the square bridge make angles $(45 + \psi)$ and $(45 - \psi)$ with the total field vector.

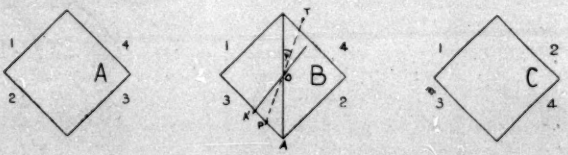

Figure 4.

Each wire has an impedance appropriate to the field in which it lies, and the unamplified current i in the detector arm of the bridge is given by

$$i = \frac{Z_1 Z_2 - Z_3 Z_4}{f(Z)} e,$$

where $Z_1 Z_2 Z_3 Z_4$ are the impedances of the bridge arms as arranged in bridge B (figure 4), e is the voltage across the input to the bridge and f is the well-known function of the impedances of the arms and of the input and detector circuits.

Let the square assembly be moved in any way and then brought back to a position in which the total field vector again lies in the line OP. Each wire will have the same impedance as before and the detector current will be the same. If the movement is constrained by the geometrical construction it will thus be possible to measure the dip.

Now suppose that the square assembly lies in the magnetic meridian plane Let the former be rotated about an axis perpendicular to its plane until a certain definite output current is shown in the indicator meter. In these circumstances, for a given value of total field and of input current, the line OP will have a fixed position relative to the assembly and will be unique.

Now the scale of the instrument reads the position A of the index, which will not be the same as the position of P.

If the assembly is rotated through 180° around OA as axis, the indicator will again show current i when the total field lies in the line OP, but the index will now be at A′, different from A, and AOP = A′OP.

Therefore the angular position of OP is the mean of the two index settings, A and A′. Evidently if the input current or total field change, both A and A′ will change, but not the value of their mean.

A bridge output current i has been considered above to illustrate the general argument. If $i=0$, we have the special case in which true balance of the bridge occurs, and which depends on the impedance and reactances of the arms fulfilling the relation $Z_1/Z_3=Z_4/Z_2=X_1/X_3=X_4/X_2$.

It is particularly to be observed, however, that in using the square bridge to find the value of the dip, it is unnecessary to attain perfect balance; any value of the bridge output current may be chosen as an arbitrary zero on which to adjust the angle of the assembly. After reversal through 180° back to front, the fresh adjustment of angle must be such as to bring back the indicator current to this arbitrary zero. Moreover, it is necessary to assume that the input current to the bridge has not changed during the two angular adjustments.

Two of the cases examined above for matched wires will now be considered for unmatched wires.

Case II. In figures 1 and 3 *a*, let D and E be the points on the sphere representing the ends of two wires. They lie on the arc ZCZ′. Also DC = CE = 45°.

Then ZD = (90 + θ − 45)° and ZE = (90 + θ + 45)°, whilst the field F_D in the wire D = $T\cos$ TD and that in the wire E = $T\cos$ TE, where T is the scalar value of the total field vector.

From the spherical triangles TZ′D and TZ′E we find

$$\left.\begin{aligned} F_D &= T\cos 45[(\cos\theta+\sin\theta)\cos i\cos\phi+(\sin\theta-\cos\theta)\sin i],\\ F_E &= T\cos 45[(\cos\theta-\sin\theta)\cos i\cos\phi+(\sin\theta+\cos\theta)\sin i]. \end{aligned}\right\} \quad \ldots\ldots(2)$$

If $\theta=i$ and $\phi=0$, equation (2) reduces to

$$F_E=F_D=T\cos 45,$$

which is case I, if the wires are perfectly matched.

Evidently, if F_D and F_E are the fields in which the arms 1 and 2, 3 and 4 lie, respectively, in the particular square bridge shown in figure 4 B, equations (2) are the expressions for these fields, in which it is observed that ϕ is the azimuth angle between the meridian plane and the plane of the assembly at balance, and that θ is greater than i and is the reading on the dip-meter scale when the bridge balances.

Now the resistance and reactance field curves for a typical nickel-iron wire are shown on page 458 of the paper (Harrison *et al.*, 1936) already referred to.

If we assume a linear relation $Z = a - mF$ between impedance Z and the field F over a small range of field near to 0·3 oersted, we may, with the help of (2), eliminate F from the above expression for Z and obtain a relation between Z and ϕ.

Thus, for four imperfectly matched wires arranged as in bridge B, figure 4:

$$Z_1 = a_1 - m_1F_1 = a_1 - m_1T\cos 45[(\cos\theta + \sin\theta)\cos i\cos\phi + (\sin\theta - \cos\theta)\sin i],$$
$$Z_2 = a_2 - m_2F_2 = a_2 - m_2T\cos 45[(\cos\theta + \sin\theta)\cos i\cos\phi + (\sin\theta - \cos\theta)\sin i],$$
$$Z_3 = a_3 - m_3F_3 = a_3 - m_3T\cos 45[(\cos\theta - \sin\theta)\cos i\cos\phi + (\sin\theta + \cos\theta)\sin i],$$
$$Z_4 = a_4 - m_4F_4 = a_4 - m_4T\cos 45[(\cos\theta - \sin\theta)\cos i\cos\phi + (\sin\theta + \cos\theta)\sin i],$$
$$\ldots\ldots(3)$$

where a_1, a_2, a_3, a_4 are the constants of the linear portions of the (field, impedance) graphs for the four wires and m_1, m_2, m_3, m_4 are their slopes.

If, as will be here assumed for simplicity, a perfect A.C. balance is possible, the condition for balance of bridge B is $Z_1Z_2 = Z_3Z_4$. Using this relation together with equations (3) we deduce the following equation between ϕ and the experimental constants of the wires:

$$\cos^2\phi[A\Sigma(mm) + k\Delta(mm)] - \cos\phi[B\Sigma(mm) + C\Delta(am) + D\Sigma(am)]$$
$$+ \Delta(aa) + E\Sigma(am) - F\Delta(am) - G\Sigma(mm) + H\Delta(mm) = 0, \quad \ldots\ldots(4)$$

where, if $t = T\cos 45°$:

$$A = t^2\sin 2\theta\cos^2 i, \qquad F = t\sin i\sin\theta,$$
$$B = 2t^2\cos 2\theta\cos i\sin i, \qquad G = t^2\sin 2\theta\sin^2 i,$$
$$C = t\cos i\cos\theta, \qquad H = t^2\sin^2 i,$$
$$D = t\cos i\sin\theta, \qquad K = t^2\cos^2 i,$$
$$E = t\sin i\cos\theta,$$

and where we define $\Sigma(mm)$ as the sum of the products of the ms in opposite arms, e.g. $\Sigma mm = m_1m_2 + m_3m_4$; $\Delta(am)$ is the difference of the products am in opposite arms, e.g. $\Delta(am) = a_1m_2 + a_2m_1 - (a_3m_4 + a_4m_3)$; $\Sigma(am)$ is the sum of the products am in opposite arms, e.g. $\Sigma(am) = a_1m_2 + a_2m_1 + a_3m_4 + a_4m_3$; $\Delta(aa)$ is the difference of the products of the as in opposite arms, e.g. $\Delta(aa) = a_1a_2 - a_3a_4$, making the convention that the cross-product in arms 1 and 2 minus that in arms 3 and 4 is positive.

On reversing the disc through 180° face to face it is noted that wires 1 and 3 change places with wires 4 and 2, so that wire 1, originally in field F_1, now lies in field F_4 and wire 3, originally in field F_3, now lies in field F_2.

Since in these circumstances $\Delta(am)$ and $\Delta(aa)$ change sign, while $\Sigma(mm)$ and $\Sigma(am)$ do not, and since the coefficients A, B, C, etc., remain unaltered, equation (4) becomes for the reversed position:

$$\cos^2\phi[A\Sigma(mm) - K\Delta(mm)] - \cos\phi[B\Sigma(mm) - C\Delta(am) + D\Sigma(am)]$$
$$- \Delta(aa) + E\Sigma(am) + F\Delta(am) - G\Sigma(mm) - H\Delta(mm) = 0. \quad \ldots\ldots(5)$$

As a numerical example, put $i = 67°$, $\theta = 67° + 10° = 77°$, T = total field = 0·46

oersted, so that $t=0{\cdot}325$. Taking the values of a and m from experimental values obtained for four actual wires used in the bridge:

for wire	I	$a_1=0{\cdot}852$,	$m_1=0{\cdot}356$,
,,	II	$a_2=0{\cdot}885$,	$m_2=0{\cdot}394$,
,,	III	$a_3=0{\cdot}827$,	$m_3=0{\cdot}260$,
,,	IV	$a_4=0{\cdot}880$,	$m_4=0{\cdot}354$.

Then equations (4) and (5) reduce to

$$0{\cdot}0023\cos^2\phi-0{\cdot}134\cos\phi+0{\cdot}063 \quad =0,$$

and

$$0{\cdot}00081\cos^2\phi-0{\cdot}1263\cos\phi+0{\cdot}0776=0,$$

the solutions of which are, respectively,

$$\cos\phi=0{\cdot}479 \text{ or } 0{\cdot}61, \text{ giving } \phi=\pm 61^\circ{\cdot}4 \text{ or } \pm 52^\circ{\cdot}4.$$

Case III. Let the disc, figure 3*c*, be rotated about its index line OC, so that it now lies in the great circle ECD.

We have $Z'T=90^\circ-i$ and $Z'C=90^\circ-\theta$.

Let E be the end of one wire; then $CE=45^\circ$, and the field in $E=T\cos TE$, where T is the total field. By consideration of the spherical triangles Z'TC and TCE it can be proved that

$$\cos TE=\cos 45\,(\cos\theta\cos i\cos\phi-\sin\phi\cos i-\sin\theta\sin i);$$

similarly for the wire D at right angles to E we have, by considering the triangles Z'TC and TCD,

$$\cos TD=\cos 45\,(\cos\theta\cos i\cos\phi+\sin\phi\cos i-\sin\theta\sin i).$$

Thus the magnetic fields in the two wires are

$$\left.\begin{aligned} F_E&=T\cos 45\,(\cos\theta\cos i\cos\phi+\sin\phi\cos i+\sin\theta\sin i),\\ F_D&=T\cos 45\,(\cos\theta\cos i\cos\phi-\sin\phi\cos i+\sin\theta\sin i).\end{aligned}\right\} \quad \ldots\ldots(6)$$

As in case II, we can obtain from equations (6) general relations expressing the condition for balance of bridge B under case III in terms of the impedance characteristics of the four wires of the bridge.

The resulting equation is

$$[A\Delta(mm)-B\Delta(am)]\cos\phi+[C\Delta(mm)]\cos 2\phi+[D\Sigma(mm)-E\Sigma(am)]\sin\phi \\ +[F\Sigma(mm)]\sin 2\phi=-[G\Delta(mm)-H\Delta(am)+\Delta(aa)], \quad \ldots\ldots(7)$$

where

$A=2T^2\cos^2 45\cos i\cos\theta\sin i\sin\theta$, $\quad E=T\cos 45\cos i$,

$B=T\cos 45\cos i\cos\theta$, $\quad F=T^2\cos^2 45\cos^2 i\cos\theta$,

$C=T^2\cos^2 45\cos^2 i\left(\dfrac{\cos^2\theta-1}{2}\right)$, $\quad G=T^2\cos^2 45\sin^2\theta\sin^2 i+E^2\left(\dfrac{\cos^2\theta+1}{2}\right)$,

$D=2T^2\cos^2 45\cos i\sin i\sin\theta$, $\quad H=T\cos 45\sin i\sin\theta$.

If we now turn the disc and assembly through 180°, so that it lies back to front, $\Delta(mm)$, $\Delta(am)$ and $\Delta(aa)$ all change sign, while $\Sigma(mm)$ and $\Sigma(am)$ do not.

The coefficients A, B, C, D, E, F, G, H remain unaltered, so the equation becomes

$$[A\Delta(mm) - B\Delta(am)]\cos\phi + [C\Delta(mm)]\cos 2\phi - [D\Sigma(mm) - E\Sigma(am)]\sin\phi - [F\Sigma(mm)]\sin 2\phi = -[G\Delta(mm) - H\Delta(am) + \Delta(aa)]. \quad \ldots\ldots(8)$$

If the sign of ϕ is changed in equation (7) we arrive at equation (8); thus the graphs of (7) and (8) are symmetrical about the Y axis, and the real solutions of (7) are equal and opposite to those of (8). Hence the mean of the two azimuth settings for balance is zero, and we eliminate the effect of want of match in the wires by reversing the disc, face to face, and taking the mean of the two azimuths as the true meridian setting.

It is to be observed that three, and only three, different arrangements of four unmatched wires in the bridge are possible. Each arrangement will give a different solution of equation (7). These three possible arrangements are shown in figure 4. The instrument as constructed is made up as in figure 4 B.

Plane of the assembly in the meridian plane

To obtain the field in the wires when the assembly lies in the meridian, we put $\phi = 0$ in equation (2) (which brings the assembly into the meridian plane). The resulting equations are:

$$F_D = T\cos 45\cos[(\theta - i) + \sin(\theta - i)],$$
$$F_E = T\cos 45\sin[(\theta - i) - \cos(\theta - i)].$$

Let ψ be the angle between the index and T. Then $\theta - i = \psi$.

Hence in bridge B

$$F_1 = T\cos 45(\cos\psi + \sin\psi) = F_2,$$
$$F_3 = T\cos 45(\cos\psi - \sin\psi) = F_4.$$

The condition for balance of this bridge being, as before, $Z_1Z_2 = Z_3Z_4$, we derive the equation

$$\Delta(am)\cos\psi + \Sigma(am)\sin\psi - t\Sigma(mm)\sin 2\psi = [\Delta(aa)]/t + t\Delta(mm), \quad \ldots\ldots(9)$$

where $t = T\cos 45°$ and $\Delta(mm)$, $\Sigma(mm)$, $\Sigma(am)$ and $\Delta(aa)$ have the same meanings as before.

If we turn the assembly through 180° about a diameter (back to front), $\Delta(mm)$ and $\Delta(aa)$ change sign and the equation becomes

$$\Delta(am)\cos\psi - \Sigma(am)\sin\psi + t\Sigma(mm)\sin 2\psi = [\Delta(aa)]/t + t\Delta(mm). \quad \ldots\ldots(10)$$

As in the previous case, changing the sign of ψ in equation (9) produces equation (10), so that the solutions of (9) and (10) are symmetrical about the Y axis and the mean of the solutions is zero, which illustrates the fact that the mean of the two index settings for balance gives the position for which the index lies in the total field, and thus gives the measure of the dip. The effect of want of match of the wires is eliminated.

C. *Construction assumed geometrically imperfect: wires assumed perfectly matched. Levelling and vertical axis errors*

Defining the *vertical axis* OF of the instrument as the normal to the azimuth

circle which passes through the centre of the disc, let OF be not in the true vertical OZ (figure 3 *d*).

If the plane ZOF makes an angle β with the meridian, and if the angle ZOF$=\gamma$, then, as before, we can find, by balancing the instrument, the plane OTF, which is such that the total field OT lies in the plane of the disc.

If θ is the value of the dip as read incorrectly in this position,

$$\text{ZOT} = 90° + i, \quad \text{FOT} = 90° + \theta.$$

Then in the spherical triangle ZTF

$$\cos(90° + \theta) = \cos(90° + i)\cos\gamma + \sin(90° + i)\sin\gamma\cos\beta.$$

(i) If $\beta = 0$, the axis lies in the plane of the meridian:

$$\cos(90° + \theta) = \cos(90° + i - \gamma),$$

or

$$\theta = i - \gamma,$$

i.e., the whole of any error in the vertical in the meridian plane appears as an error in the value of the dip, which is otherwise obvious.

(ii) If $\beta = 90°$,

$$\cos(90° + \theta) = \cos(90° + i)\cos\gamma,$$

or

$$\sin\theta = \sin i \cos\gamma.$$

i.e., the level in the transverse direction has much less effect than in the plane of the meridian; 1° error in this plane gives 1′ error in the result.

The error which is found if the vertical axis of the instrument is not perpendicular to the base plane will be a constant of the instrument.

Let θ be the reading for the setting in which the plane AOD makes an angle β with the meridian (figure 3 *d*). Then

$$\cos(90° + \theta) = \cos(90° + i)\cos\gamma + \sin(90° + i)\sin\gamma\cos\beta.$$

To rotate through 180° we replace β by $(\beta + 180)°$. Then

$$\cos(90° + \theta') = \cos(90° + i)\cos\gamma - \sin(90° + i)\sin\gamma\cos\beta.$$

If β be small,

$$\cos(90° + \theta) = \cos(90° + i - \gamma), \qquad \cos(90° + \theta') = \cos(90° + i + \gamma),$$

and, therefore, $\frac{1}{2}(\theta + \theta') = i$.

Thus the error can be eliminated by rotating the instrument 180° in azimuth and taking a mean of the dip readings.

The error of levelling cannot be corrected by any rotation within the instrument, since this depends on the direction of the vertical, which can only be determined by other methods. The dip indicator will *show* the direction of the total field correctly, but the expression of this as an angular deviation from the horizontal depends on our knowledge of the horizontal. Thus ultimately the accuracy of the instrument depends on that of the bubble.

Error in dip measurement due to error in meridian setting

When the assembly is in a vertical plane, if i is the dip, ϕ the angle between the plane of the disc and the magnetic plane and ψ the angle between the index line and the total field vector, then $\theta = i + \psi$ and equation (1) is seen to be the well known relation

$$\cos\phi = \cot(i + \psi)\tan i.$$

From this it is found that 2° error in azimuth setting gives less than 1′ error in the dip.

§ 5. EFFECT OF HYSTERESIS

It has been shown (Harrison *et al.*, 1936) that a small hysteresis exists in the (impedance, field) relation for the nickel-iron wires; it is necessary to examine whether or not this has any effect on the measurement of dip.

The use of the instrument involves three different kinds of movement:

(i) Rotation round the axis AC.

(ii) The move from the right-hand to the left-hand end of the scale.

(iii) Rotation in azimuth.

(i) If the index is in the total field the wires will remain in the same field throughout movement (i). There will, therefore, be no hysteresis effect. If, as is shown in figure 4 B, the index OA makes an angle ψ with the total field, the wires will move through fields which vary in value between $T\cos(45°+\psi)$ and $T\cos(45°-\psi)$. In a well matched bridge ψ is found to be less than 2°, and the difference caused by hysteresis under movement (i) is negligible.

Since no assumption has been made as to the magnetic condition of the wires except that it remains constant between a pair of readings, these two readings will determine the dip.

(ii) During the movement from one end of the scale to the other, and the rotation of 180° in azimuth, the wires pass through much larger changes of field.

It has been shown that wires D and E (figure 3 *a*) lie in fields $(\theta-45°)$ and $(\theta+45°)$ respectively. Then when the disc has been moved to the position in which $\theta=i+45°$, D and E will lie in fields making angles i and $i+90°$ with the horizontal, i.e., D is in the total field, which is the maximum possible field. E lies perpendicular to it in the zero field. Therefore, when the disc passes through this position to that for determining the dip, wire E will be in the condition represented by the upward branch of the hysteresis loop and wire D in that represented by the downward one. If, before making this movement, the disc is changed face to face, the wires D and E interchange their positions and E now approaches the balance position from the maximum and D from the minimum.

(iii) Similarly it can be shown that during the rotation in azimuth, wire D (figure 3 *a*) passes through a zero field and wire E into a maximum field. If the disc is rotated about its index before the movement in azimuth, D and E, as before, will change place, and E will pass through the zero field.

Observations of the value of the dip in the laboratory have been taken after the disc had been moved in each of the above ways.

The following are typical results:—

	θ_0	θ_1	Mean value $=i$	Difference $=2\psi$
Moved in meridian plane, face E:	65° 18′	65° 30′	65° 24′	12′
Moved in meridian plane, face W:	65° 59′	64° 51′	65° 25′	68′

Moved in azimuth, face E:

θ_0	θ_1	Mean value $=i$	Difference $=2\psi$
65° 54′	64° 54′	65° 24′	60′

Moved in azimuth, face W:

65° 35′	65° 14′	65° 24′·5	21′

They show that whereas the hysteresis can change the position of the magnetic axis by as much as 1°, the value of the dip is unaffected by this.

§6. THE ELECTRICAL UNIT

The circuit used in association with the dip meter described in this paper is shown in figure 5.

It is essentially the same as that previously described, except that the low-pass filter has been replaced by a tuned-input transformer to the amplifier which prohibits the passage into the latter of any harmonics. It has been shown (Harrison and Rowe, 1938) that if $\pm E$ is the polarizing voltage on the

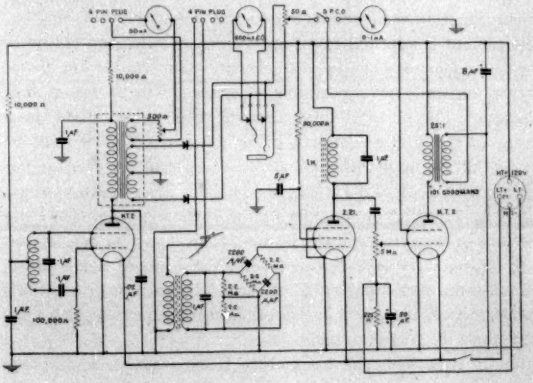

Figure 5.

rectifier bridge, and if the output voltage from the magnetic bridge is given by $e = e_0 \sin(\omega t + \alpha)$, where α is the phase difference between E and e, then the difference in amplitude between the vectors $(E+e)$ and $(-E+e)$ is proportional to $2e_0 \cos\alpha$, which is proportional to the deflection of the indicator meter. Thus, to obtain the maximum indicator deflection for a given bridge output, we must make $\cos\alpha$ a maximum or $\alpha = 0$.

In order to ensure that this phase relation shall hold between the magnetic-bridge output voltage and the polarizing voltage on the rectifier bridge, a phase shifter is associated with the circuit of the tuned-input transformer, which performs the necessary phase adjustment between E and e.

As has been pointed out in §4, B, it is an interesting feature of the square bridge that balancing is a process which need not consist of adjusting the

assembly to give *zero* reading on the indicating meter; any scale division on the meter may be chosen as arbitrary zero, and provided the electrical conditions remain constant between the readings on the dip meter of two balance positions (disc face to face), the mean of these two readings will give the dip. This is due to the fact that after rotation of the disc (face to face) each wire must necessarily, at "balance", lie in the same field as it did before rotation.

§7. EXPERIMENTAL

(a) *Determination of the magnetic meridian*

Reference to §4(A) shows that the meridian may be determined in either of two ways, described as case II or case III. Using the method of case II, the disc is rotated until its plane lies in the plane of the ring R, when the plunger P engages and holds the disc in this position.

The angle θ is made greater than i, which may be assumed to be known very roughly at the place of the experiment. For example, if the dip is 67° or thereabouts, the drum is turned until the fiducial mark corresponds with some angle greater than 67°, say 73°.

The instrument is then connected to the electrical unit and is levelled. The bridge current is adjusted to 40 ma. (which is twice the optimum current for each wire) and meter C is put into circuit. On rotating the instrument in azimuth, two, and only two, positions, ϕ_1 and ϕ_2, will be found at which the bridge balances. The mean of these azimuth readings, ϕ_m, gives the meridian, the direction of the magnetic north being determined at the same time by remembering that the index line AC slopes *downwards towards the north*.

Using the method of case III, the disc having been located perpendicular to the ring R by engaging the plunger in one of the extension arms, the instrument is rotated in azimuth until the bridge balances. The disc is now rotated in the ring R about its axis, through 180°, the plunger again engaging and holding the disc in its new position. A second balance position is read on the azimuth scale. As in the previous case, the mean of the two readings gives ϕ_m, which determines the meridian line.

It should be remarked that, unlike case II, this way of finding the line of intersection of the meridian plane with the horizontal does not distinguish between north and south, since there are four balance positions, one pair for each reversal of the disc, as we rotate in azimuth through 360°.

(b) *Determination of the dip*

The apparatus having been set in azimuth at the angle ϕ_m, using either of the two ways of finding the meridian, the micrometer head is turned until the indicator meter shows an arbitrary output current at an angular reading θ_1 on the right-hand half of the scale S. The disc is then rotated about its axis through 180°, when the plunger again engages with the disc and holds it. A new setting of the disc corresponding to the same arbitrary meter "zero" is thus found. Let the reading be θ_2.

In order to eliminate the vertical error of the instrument, the disc is now turned in its own plane by turning the micrometer head until the fiducial mark

is opposite the left-hand portion of the divided scale S. The azimuth setting is then changed by 180°.

The angular reading θ_3 on S corresponding to the same (arbitrary) zero on the D.C. indicator meter is noted. Reversing the disc and adjusting the micrometer head until the indicator meter reading is again at the arbitrary zero, a fourth angular reading, θ_4, is obtained on S.

The dip is given by the mean of θ_1, θ_2, θ_3 and θ_4. By this procedure we eliminate errors due to want of match of the wires, instrumental errors due to inaccurate geometric mounting of the wires and instrumental errors due to the vertical axis not being normal to the base plane.

The procedure described above for the measurement of the dip is simplified from an observational point of view by using as arbitrary zero the central

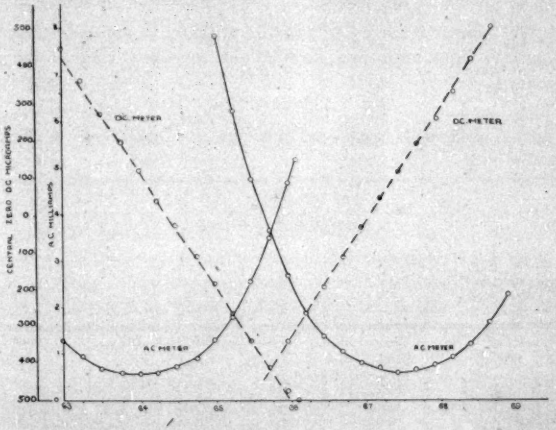

Figure 6.

graduation on the central-zero microammeter which deals with the rectifier bridge output.

In order to illustrate the effect of using either the A.C. or the D.C. meter as indicator, a comparison was made of their readings for different settings of the square bridge in the meridian plane. After reversal of the disc back to front, another series of readings was obtained. These observations are shown plotted in figure 6.

It is found that both the A.C. and the D.C. readings give curves which are symmetrical about the line $x = 65° 43'$. This is the value of the dip in the laboratory from the curves. It can be found by observing the point at which the two curves cross, or by taking the mean values of x for any arbitrary value of y.

The A.C. meter is very insensitive at its minimum; moreover, it does not distinguish on which side of the total field the index of the unbalanced bridge

lies. Thus there will be four positions on the dip scale for each reading of the A.C. meter and only two for each reading of the D.C. meter, which gives two straight lines ever the whole range.

(c) *Value of the dip*

Many measurements of the dip have been made in a wooden shed in the country near Chigwell, Essex. A typical series of results is given in the table.

Left-hand end of scale 10.50 a.m., 17 May

θ_3	θ_4	Mean of θ_3 and θ_4	Deviation from mean
° ′	° ′	° ′	′
67 17·8	67 13·0	67 15·4	+0·4
67 18·5	67 12·0	67 15·25	+0·25
67 19·5	67 14·5	67 17·0	+2·0
67 18·9	67 10·0	67 14·45	−0·55
67 18·1	67 14·3	67 16·2	+1·2
67 17·0	67 16·0	67 16·5	+1·5
67 16·2	67 11·3	67 13·75	−1·25
67 17·0	67 12·3	67 14·65	−0·35
67 15·5	67 8·4	67 11·95	−3·05
67 15·4	67 11·6	67 13·5	−1·5
67 16·0	67 11·3	67 13·65	−1·35
67 16·7	67 11·7	67 14·2	−0·8
	11.30 a.m.		
° ′	° ′	° ′	′
66 55·0	67 34·0	67 14·5	−0·5
66 57·0	67 35·0	67 16·0	+1·0
66 57·0	67 32·5	67 14·75	−0·25
66 55·7	67 31·3	67 13·5	−1·5
66 56·0	67 34·0	67 15·0	0
66 55·5	67 36·0	67 15·75	+0·75
66 56·2	67 32·0	67 14·1	−0·9
66 55·9	67 32·1	67 14·0	−1·0
66 56·7	67 33·4	67 15·05	+0·05
66 57·1	67 31·0	67 14·05	−0·95
66 58·5	67 33·0	67 15·75	+0·75
66 59·0	67 34·0	67 16·5	+1·5
67 0·5	67 33·1	67 16·8	+1·8
66 59·0	67 34·7	67 16·85	+1·85
Right-hand end of scale	11.10 a.m.		
° ′	° ′	° ′	′
67 0	67 33·0	67 16·5	+0·3
66 58·3	67 32·9	67 15·6	−0·6
67 0·1	67 32·8	67 16·45	+0·25
66 59·0	67 32·0	67 15·5	−0·7
66 59·0	67 31·8	67 15·4	−0·8
67 0·1	67 33·1	67 16·6	+0·4
66 59·0	67 34·0	67 16·5	+0·3
66 59·2	67 33·9	67 16·55	+0·35
67 0·5	67 33·8	67 17·15	+0·95
67 1·0	67 32·0	67 16·5	+0·3
66 58·7	67 32·1	67 15·4	−0·8
66 59·5	67 33·1	67 16·3	+0·1

Arithmetic mean of the right-hand values of the dip = 67° 16′·2.
Probable error of one observation at the right-hand end = 0′·385 = 23″.
Probable error of mean = 6″·5.
Arithmetic mean of the left-hand values of the dip = 67° 15′.
Probable error of one observation at the left-hand end of scale = 0′·86 = 52″.
Probable error of the mean = 10″.

Value of dip in the wooden shed = 67° 15′ 36″.

The difference between the mean value obtained on right and left ends of scale is seen to be 1′·2, which gives as the vertical error of the instrument 30″.

The value of the dip as measured at Chigwell could not be checked, as the dip in this locality was known only by interpolation and was subject to an uncertainty of 10 minutes of arc.

Consequently it was arranged, by courtesy of the Superintendent of Abinger Magnetic Observatory, to measure the dip there on 12 May 1943. Taking mean values of eight readings made between 3.30 p.m. and 4.0 p.m., the new instrument gave

66° 44′ 28″.

The mean of the values over the same time interval supplied by the Abinger Observatory was

66° 44′ 30″.

ACKNOWLEDGMENTS

We wish to acknowledge, with thanks, assistance received from Mr. P. Gasper, B.Sc.(Eng.), in the design of the electrical unit, especially of the tuned-input transformer to the amplifier. Our thanks are also due to Messrs. Henry Hughes and Son, Ltd., for permission to publish this paper.

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A NOTE ON THE POLARIZATION OF A PLANE POLARIZED WAVE AFTER TRANSMISSION THROUGH A SYSTEM OF CENTRED REFRACTING SURFACES, AND SOME EFFECTS AT THE FOCUS

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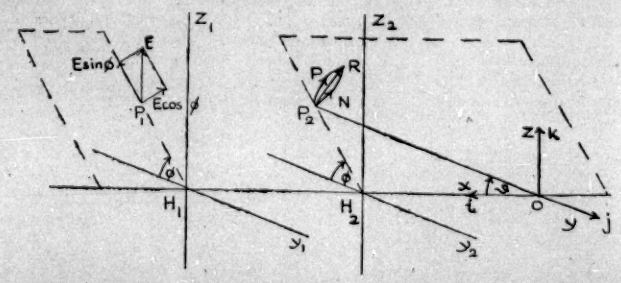
MS. received 8 *July* 1943

ABSTRACT. It is the purpose of this note to consider the state of polarization of a wave emerging from a lens system in the light of Fresnel's Laws, when the incident wave is plane-polarized and proceeds from a point on the axis at a great distance from the system. Three special cases can be distinguished: the one in which θ, the semi-angular aperture of the system, is small, whereas the angles of incidence within the system are large; the second, in which θ is comparatively large, whereas the transmission of energy along marginal ray-paths can be regarded as complete; the third, in which both θ and the angles of incidence are small.

In the course of the discussion an assumption concerning the planes containing the light vectors made in a previous communication is shown to be not strictly valid. The approximation, however, is shown to be very close.

THE unit surfaces of the system $z_1H_1y_1$, $z_2H_2y_2$ (see figure) are regarded as plane for the purpose of this treatment. It will be seen that this restriction is of no consequence, since any departure from it merely alters the absolute values of the components of the light vector by the same factor, the direction of the resultant remaining the same. The existence and magnitude



of this effect in a system fulfilling the sine condition have been considered elsewhere (see "The Airy Disc Formula for Systems of High Relative Aperture" Hopkins, 1943, *Proc. Phys. Soc.* **55**, 116.)

H_1H_2O is the axis of the system, O being the focus of a wave incident from the left. This wave is assumed to be plane, plane-polarized and of uniform

amplitude. The Oz direction is taken to be that of the light vectors in the incident wave. A portion of the wave at P_1 (ρ, ϕ) in the plane $z_1H_1y_1$ is imaged at P_2 (ρ, ϕ) in the plane $z_2H_2y_2$, whence it proceeds along the ray-path P_2O to the focus at O. The angle $\widehat{H_2OP_2}$ is θ.

The light vector at P_1 is P_1E, having components $E\sin\phi$ and $E\cos\phi$ in directions parallel with and normal to the plane of incidence respectively. The former will remain in the place of incidence—the meridional plane $P_1H_1H_2P_2$—and emerge as P_2P, which is perpendicular to P_2O. Similarly the component normal to the plane of incidence will emerge as P_2N, which is also normal to the plane of incidence, and, therefore, to P_2O. The magnitudes of these components can be determined by repeated application of the appropriate Fresnel formulae.

If there are λ refracting surfaces, then along a ray-path, characteristic of an annulus $\rho =$ constant, let the successive angles of incidence and emergence be

$$i_1 i_2 i_3 \ldots\ldots\ldots\ldots i_\lambda,$$
$$e_1 e_2 e_3 \ldots\ldots\ldots\ldots e_\lambda.$$

The magnitude of P_2P is then

$$E\sin\phi \cdot \frac{2\cos i_1 \cdot \sin e_1}{\sin(i_1+e_1)\cos(i_1-e_1)} \cdot \frac{2\cos i_2 \cdot \sin e_2}{\sin(i_2+e_2)\cos(i_2-e_2)} \cdots\cdots \frac{2\cos i_\lambda \cos e_\lambda}{\sin(i_\lambda+\rho_\lambda)\cos(i_\lambda-e_\lambda)} \quad \ldots\ldots(1)$$

$$= P(\rho^2)E\sin\phi, \quad \ldots\ldots(2)$$

where $P(\rho^2)$ is a function expressing the variation of the particular transmitted amplitude with ρ.

In an identical manner the magnitude of P_2N is

$$E\cos\phi \cdot \frac{2\cos i_1 \sin e_1}{\sin(i_1+e_1)} \cdot \frac{2\cos i_2 \sin e_2}{\sin(i_2+e_2)} \cdots\cdots \frac{2\cos i_\lambda \sin e_\lambda}{\sin(i_\lambda+e_\lambda)} \cdot \quad \ldots\ldots(3)$$

$$= N(\rho^2)E\cos\phi. \quad \ldots\ldots(4)$$

It will be seen that

$$N(\rho^2) = P(\rho^2) \prod_{r=1}^{\lambda} \cos(i_r - e_r). \quad \ldots\ldots(5)$$

If it is assumed that the transmission of energy is completed, then we must write

$$P(\rho^2) = N(\rho^2) = 1/\cos^{\frac{1}{2}}\theta, \quad \ldots\ldots(6)$$

for the energy entering through an area S_1 at P_1 emerges through an equal area S_2 at P_2, where the normal to S_2 makes an angle θ with the direction of propagation P_2O. The ratio of the fluxes along the emergent and incident ray-paths is thus equal to $1/\cos\theta$ and, assuming the same medium on either side of the system, the amplitudes are in the ratio $1/\cos^{\frac{1}{2}}\theta$.

Let $\mathbf{N}$ be a unit vector normal to the plane of incidence. Then, if $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is a system of unit vectors in the x, y, and z directions, $\mathbf{N}$ is given by

$$\mathbf{N} = \sin\phi\mathbf{j} + \cos\phi\mathbf{k}. \quad \ldots\ldots(7)$$

If a unit vector in the direction of OP_2 is written $\mathbf{V}$, then

$$\mathbf{V} = \cos\phi\mathbf{i} - \sin\theta\cos\phi\mathbf{j} + \sin\theta\sin\phi\mathbf{k},$$

and a unit vector in the direction P_2P is given by

$$\mathbf{P} = \mathbf{V} \times \mathbf{N},$$

where the cross indicates a vector product, then

$$\mathbf{P} = -\sin\theta\mathbf{i} - \cos\theta\cos\phi\mathbf{j} + \cos\theta\sin\phi\mathbf{k}. \quad \ldots\ldots(8)$$

The resultant light vector is now given in terms of the amplitudes (2) and (4) and the unit vectors (7) and (8). It is

$$\begin{aligned}\mathbf{R} &= E\cos\phi N(\rho^2)\mathbf{N} + E\sin\phi P(\rho^2)\mathbf{P} \\ &= N(\rho^2)E\{\sin\phi\cos\phi\mathbf{j} + \cos^2\phi\mathbf{k}\} + \\ &\quad P(\rho^2)E\{-\sin\theta\sin\phi\mathbf{i} - \cos\theta\cos\phi\sin\phi\mathbf{j} + \cos\theta\sin^2\phi\mathbf{k}\}. \quad \ldots\ldots(9)\end{aligned}$$

If ϕ take the values $\phi, \pi-\phi, \pi+\phi, 2\pi-\phi$ successively while θ remains constant, then $N(\rho^2)$, $P(\rho^2)$ and E remain constant, and the terms in $\sin\phi$ and $\sin\phi\cos\phi$ cancel in pairs, leaving to be considered, so far as the effects near O are concerned, only the z-component of $\mathbf{R}$. This is

$$(\mathbf{R})_z = E\{N(\rho^2)\cos^2\phi + P(\rho^2)\cos\theta\sin^2\phi\}. \quad \ldots\ldots(10)$$

First, suppose θ small, that is $\cos\theta \doteq 1$, then (10) becomes

$$\begin{aligned}(\mathbf{R})_z &= E\{N(\rho^2)\cos^2\phi + P(\rho^2)\sin^2\phi\} \\ &= P(\rho^2)E\{\sin^2\phi + \cos^2\phi\prod_{r=1}^{\lambda}\cos(i_r - e_r)\}. \quad \ldots\ldots(11)\end{aligned}$$

If, furthermore, the angles of incidence are small, the angles i_r, e_r are also small; and we have for the effective light vector the convenient formula

$$(\mathbf{R}) = P(\rho^2)E. \quad \ldots\ldots(12)$$

The resultant light vector in the emergent wave lies in a plane perpendicular to the ray-path P_2O. If ψ is the angle it makes with the normal to the plane of incidence, then from (2) and (4)

$$\begin{aligned}\tan\psi &= \frac{P(\rho^2)}{N(\rho^2)}\tan\phi, \\ \cot\psi &= \cot\phi\prod_{r=1}^{\lambda}\cos(i_r - e_r). \quad \ldots\ldots(13)\end{aligned}$$

In the special case when i_r and e_r are small throughout the system, we have

$$\psi = \phi. \quad \ldots\ldots(14)$$

If, now, we consider cases where θ is large, but in which i_r and e_r are small, the direction of the light vector is again given by (14). The effective amplitude of the vector at the focus is, from (6) and (10),

$$(\mathbf{R})_z = \frac{E}{\cos^{\frac{1}{2}}\theta}\{\cos^2\phi + \cos\theta\sin^2\phi\}. \quad \ldots\ldots(15)$$

In the communication mentioned above, the term $E/\cos^{\frac{1}{2}}\theta$ was contained in a factor expressing the amplitude of the wave at points on the spherical cap

comprising the emergent wave front in terms of the uniform amplitude of the incident wave. The light vector was assumed to be perpendicular to the emergent ray-path, and in a plane containing this latter and perpendicular to the xOy-plane. This leads to a factor

$$\{1 - \sin^2\theta \sin^2\phi\}^{\frac{1}{2}}, \qquad \ldots\ldots(16)$$

instead of that contained in the bracket of (15). The integrations effected in the earlier paper present, in fact, less difficulty if the latter is employed. It will be shown, however, that the approximation is very close for the values of θ considered.

Firstly, it will be seen that the two factors are identical when $\phi = 0$, $\pi/2$, π and $3\pi/2$. To consider intermediate values, write

$$\begin{aligned} &\cos^2\phi + \cos\theta \sin^2\phi \\ &\quad = 1 - (1 - \cos\theta)\sin^2\phi \\ &\quad = 1 - 2\sin^2 \tfrac{1}{2}\theta \,.\, \sin^2\phi, \end{aligned} \qquad \ldots\ldots(17)$$

and again,

$$\begin{aligned} &\{1 - \sin^2\theta \sin^2\phi\}^{\frac{1}{2}} \\ &\quad = 1 - \tfrac{1}{2}\sin^2\theta \sin^2\phi - \tfrac{1}{8}\sin^4\theta \sin^4\phi - \ldots. \\ &\quad = 1 - 2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\sin^2\phi - \tfrac{1}{8}\sin^4\theta \sin^4\phi - \ldots. \\ &\quad = 1 - 2\sin^2\frac{\theta}{2}\sin^2\phi + 2\sin^4\frac{\theta}{2}\sin^2\phi - \tfrac{1}{8}\sin^4\theta \sin^4\phi - \ldots. \end{aligned} \qquad \ldots\ldots(18)$$

The difference between (18) and (17) is

$$\begin{aligned} &2\sin^4\frac{\theta}{2}\sin^2\phi - \tfrac{1}{8}\sin^4\theta \sin^4\phi - \ldots. \\ &\quad = 2\sin^4\frac{\theta}{2}\sin^2\phi - 2\sin^4\frac{\theta}{2}\sin^4\phi + 2\sin^8\frac{\theta}{2}\sin^4\phi - \ldots. \\ &\quad = 2\sin^4\frac{\theta}{2}\sin^2\phi\cos^2\phi + 2\sin^8\frac{\theta}{2}\sin^4\phi - \text{high powers of } \sin\theta \sin\phi, \end{aligned}$$

which is very small for $\theta < 45°$.

It is perhaps worthy of note that (10) contains the interesting suggestion that near the focal plane the light disturbance is plane polarized in a direction identical with that of the incident wave, no matter what the aperture or the conditions of transmission.

Finally, the extension of this treatment to include cases where the incident wave is not plane is immediately suggested, but is of trivial significance.

X-RAY STUDY OF THE HYSTERESIS EFFECT OBSERVED IN THE PALLADIUM-HYDROGEN SYSTEM

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ABSTRACT. The hysteresis effect observed in the palladium-hydrogen system when the pressure is varied has been investigated by the x-ray method at different temperatures ranging from 60° c. to 130° c. The effect of charging the specimens with hydrogen for long periods was also examined. Definite relations were observed between the temperature and certain critical values of the pressure connected with the appearance and disappearance of the two phases which are present in the system. The α-phase lines in the structure spectra were initially sharply defined, but the β-phase lines were always diffuse, and so also were the α-phase lines when the phase reappeared again on decreasing the pressure of the gas.

The behaviour of charged palladium under various conditions of experiment in the region of pressure where hysteresis occurs is found to be remarkably definite, although the system is not in its final state of equilibrium.

§1. INTRODUCTION

THIS paper* contains an account of the continuation of work which was conducted by the x-ray method on the occlusion of hydrogen by palladium, published in two previous papers (Owen and Jones, 1937a and b). The work described in those papers required extension to cover a wider range of temperature, and certain points were mentioned which needed further investigation, in particular the effect of prolonged charging of the specimens with hydrogen, and the factors governing the appearance and disappearance of the two phases existing in the system, both with increasing and decreasing pressures, with a view to gaining further information concerning the hysteresis phenomenon observed in the system when the pressure is varied. To carry out the investigation satisfactorily it was necessary to modify the apparatus previously employed so that several photographs could be taken over long periods during which the temperature and the pressure of the gas were maintained at steady values; it was also considered desirable to be able to keep the specimen, maintained at a given temperature (or pressure), in the same enclosure whilst the pressure (or temperature) assumed different values. The modification introduced in the construction of the apparatus allowed these conditions to be realized.

§2. EXPERIMENTAL

In the present investigation the fibre camera was exclusively employed and the experimental arrangement was different from that formerly adopted.

* The work described in this paper was completed before the outbreak of war.

Whereas previously the camera, which was of the focusing type, was mounted in the gas enclosure, necessitating the recharging of the enclosure for each exposure, for the present work the camera was mounted outside the gas enclosure so that the specimen could be left undisturbed in the hydrogen atmosphere and several exposures made under the same or different conditions of temperature and pressure within the enclosure.

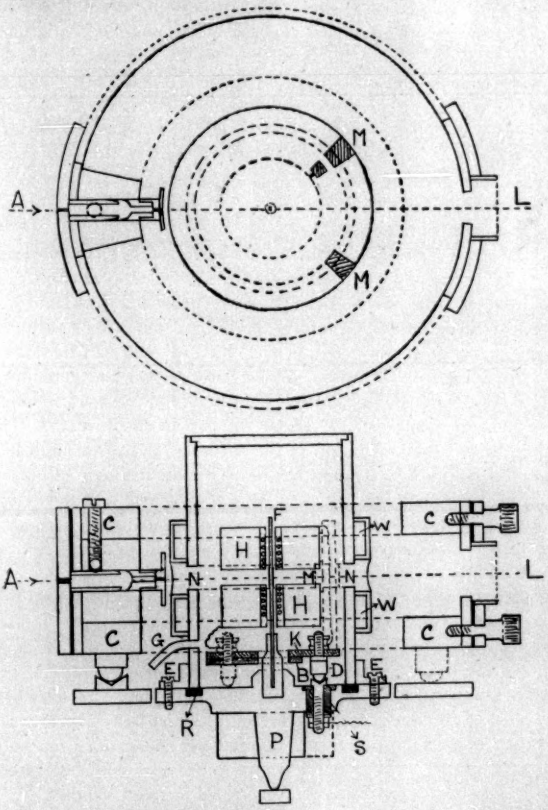

Figure 1.

The apparatus is shown in vertical and horizontal section through the centre of the specimen in figure 1. It consists of a small cylindrical chamber, about 7 cm. high and 5 cm. internal diameter, enclosing the specimen F and heater H. The chamber rests on a well-greased rubber washer R fitting a groove cut in a circular brass base-plate which can be oscillated in the cone-bearing P. Additional pressure on the rubber washer is produced by screwing the brackets

E to the base-plate, thus ensuring a vacuum-tight joint. The chamber is evacuated through the side tube G and, after evacuation, current is passed through the heater to drive gas out of the material. This is continued until no change in pressure can be observed on further heating, when the chamber is disconnected from the pump. The camera C is mounted on three legs furnished with steel balls, two of which rest respectively in a hole and a groove, and the third on the plane surface of a fixed plate surrounding the movable base-plate carrying the chamber. The "hole, slot and plane method" is also employed for mounting the heater inside the chamber. By this method the heater and the camera always take up the same position relative to each other and the fibre. The heater is mounted on a steatite base K supported by the brass plate B, and the leg D rests in the circular hole cut at the end of a rod which passes through the steatite plug S. Current is supplied to the heater through this rod, the other terminal being connected to the base-plate, which is earthed. For the small currents employed, the above method of leading the current in and out of the heater proved satisfactory.

A slot NN, about 7 mm. wide, was cut in the wall of the chamber, leaving only the supports MM to hold the two halves together; a layer of cellophane, 0·05 mm. thick, mounted on the water channels WW, proved a satisfactory window, which, after careful selection of a suitable piece, was found to be vacuum tight.

The camera had an effective radius of 55·63 mm., and had six fiducial marks cut in its circumference. It was of robust construction and carefully levelled, so that the x-ray beam passing through the slit system fell at right angles to the fibre specimen and symmetrically across the gap in the furnace. The beam, after leaving the cellophane window of the chamber, passed through an aperture in the camera large enough to allow the beam to pass through without touching the sides; this precaution was taken to reduce scattering and thus yield clearer photographs. The fibre when oscillating was examined with a reading microscope furnished with a micrometer scale, and it was adjusted until no appreciable wobble occurred.

The heater was calibrated, in a manner similar to that described previously, both *in vacuo* and in hydrogen, at different pressures (Owen and Jones, 1937 a). It was not necessary to know the temperature to a high degree of accuracy, but it was essential that it should remain constant for each set of experimental conditions. This was achieved by accurately adjusting the current through the heater and maintaining constant pressure, which was checked on a glass pressure-gauge furnished with a long pointer and constructed on the principle of the Bourdon gauge. Very small changes in pressure could be detected with this gauge. The whole apparatus was free from mercury vapour.

Hydrogen was prepared in the way previously described (Owen and Jones, 1937 a). Rubber connections were reduced to a minimum, two joints only being necessary, one to the metal Bourdon gauge and the other to the chamber so as to allow it to oscillate. The gas was thoroughly dried by passing it through drying tubes and then allowing it to remain in contact for several hours, and sometimes days, with phosphorus pentoxide contained in two large bottles. On its way to the chamber it was passed over more phosphorus pentoxide, and through a tube containing caustic potash. That the gas was dry on entering

the chamber was evident from the fact that the phosphorus pentoxide in the two reservoirs was not renewed during three years' experimenting, and at the end of that period it bore only slight traces of the effect of moisture.

The lattice parameter of palladium was taken to be $3{\cdot}8825_6 \pm 0{\cdot}0003$ A. at 20° C., this being the value recorded by Owen and Jones (1937 a); their value of the thermal expansion of palladium was also used. The element was not less than 99·9% pure. After filing, the material was sieved through a 250 mesh and annealed *in vacuo* for about four hours at 450° C. to remove the distortion produced by the filing operation. This grade of particle was used to cover the fibre mounted in the fibre camera. Each fibre specimen was used several times over, so that, after being charged with hydrogen, it had to be degassed and annealed at about 450° C. to bring it back to its original condition.

Three methods of charging the specimen with hydrogen were previously employed, the best of these being that in which the specimen was heated *in vacuo* to a temperature about 20° C. higher than the final temperature required, and maintained at this temperature while the gas was admitted in small volumes until the required pressure was reached. The temperature was then brought to its final value. This method was also employed on the present work.

The evacuated chamber was thoroughly tested for leaks after letting it stand for some time to allow the phosphorus pentoxide in the tube leading to the chamber, and also in the chamber itself, to absorb all the moisture in the apparatus. The length of time required for this to take place depended upon the length of time for which the chamber had previously been open to the atmosphere. For this reason the chamber, even when not in use, was kept evacuated. After ascertaining that the chamber was dry and leak-proof, the current was passed through the heater, the pressure gauge being kept constantly under observation as the temperature rose to its final value to check that no gas was emitted by the heater. After the temperature had reached a steady value the hydrogen was slowly admitted.

§ 3. RESULTS

(i) *Hysteresis curves at* 100° C.

A careful study by the x-ray method has already been made (Owen and Jones, 1937 a) of the hysteresis effect observed in the palladium-hydrogen system when the specimen is maintained at 100° C. and the pressure is progressively increased from a low value up to atmospheric and again when it is decreased from atmospheric to low values. It was then found that with ascending pressure the second phase (β), which has the larger parameter, appeared at about 34 cm., and that the α-phase disappeared at about 45 cm. Hg pressure, whereas with descending pressure the α-phase did not reappear until the pressure had reached 21 cm. of mercury, and the β-phase did not disappear until the pressure had been reduced to 18 cm. of mercury.

It was considered desirable, as a first step with the newly designed apparatus used in the present investigation, to follow the procedure previously adopted in order to find whether the previous results could be repeated. Accordingly, for the ascending isothermal the specimen was maintained at 100° C. and the pressure gradually raised to a given value and maintained at this value for

two hours before exposing to the x-ray beam. For the descending isothermal, the pressure was initially kept at 66·0 cm. for two hours and then decreased gradually to the pressure at whch the exposure was to be made, and maintained at this value for two hours before making the exposure. After each exposure the specimen was degassed and the same procedure as above again repeated for each pressure.

The curves now obtained were in all essentials similar to those already published, and the deductions made from them were the same as before, namely, that only those parts of the ascending and descending isothermals, where a single component exists, could be retraced by altering the pressure, and that this reversibility disappears immediately the second component is formed. Also, up to the pressure at which the α-phase disappears, the lines in the structure-spectra remain sharp, the $K\alpha_1$ and $K\alpha_2$ lines being well resolved. The β-phase lines are, however, diffuse on their first appearance, and all attempts to obtain them sharply defined failed. On the descending isothermal, the β-phase lines were still diffuse, and when the α-phase made its reappearance its lines were even more diffuse than those of the β-phase. This diffuseness of the α-phase lines remained even when the enclosure was highly evacuated. It was necessary to submit the specimen to the heat treatment referred to earlier in order to restore the lines to their original sharpness of definition.

Owing to the difference in definition of the lines, the accuracy of measurement is greatest for the ascending isothermal in the α-phase, where the lines are well defined; the unresolved lines of the β-phase yield less accurate results, the accuracy claimed being ±0·004 A., whilst the measurements made on the α-phase on the descending isothermal are still less accurate on account of the very diffuse nature of the lines. Within the limits of experimental error, and over the same range of pressure in each pure phase, the ascending and descending isothermals follow the same curves.

The isothermals at 100° C. were redetermined by adopting a different procedure, so as to make full use of the special advantages offered by the apparatus designed for this investigation. Since the camera could be removed without disturbing the chamber it was not necessary to degas the specimen after each exposure, and so the isothermals at 100° C. were obtained when the specimen was charged continuously without degassing, any particular pressure being maintained at a steady value for two hours before an exposure. After each exposure the pressure was changed, maintained at this new value for two hours, and then another exposure was made, so that the whole operation took much less time than it did previously. When the highest pressure on the ascending isothermal was reached, the pressure was then reduced in stages in the same manner, exactly as it was increased for the ascending isothermal; there was no recharging to 66·0 cm. pressure after each exposure followed by a reduction of the pressure to the required value.

As the isothermal could not be determined in one day, the specimen had to be left in an atmosphere of hydrogen at some particular pressure overnight, and in the morning the pressure was changed and maintained at its new value for two hours before making an exposure. This long interval of time at one pressure could not be avoided, but experience showed that its effect was not different

from that of many consecutive two-hour periods of charging at different pressures.

It was sometimes necessary to switch off the heater current for several hours, and on these occasions the specimen was removed from the chamber and the observations continued with either the degassed specimen or an entirely new specimen. In order that the specimen might attain a state similar to that obtaining when the heater was switched off it was left in an atmosphere of hydrogen for a few hours and then the pressure was adjusted to and maintained for two hours at the value required. The isothermals obtained in this way agreed closely with those found with the other method of procedure.

Table 1. Phase changes at 100° c.

Pressure (cm. Hg)	Parameter	
	α-phase	β-phase
Increasing pressure		
0	3·8868	—
10	3·8898	—
20	3·8919	—
27	3·8934	4·0192
30	3·8940	4·0204
40	3·8958	4·0233
41	3·8960	4·0234
50	—	4·0242
60	—	4·0249
70	—	4·0254
Decreasing pressure		
70	—	4·0250
60	—	4·0246
50	—	4·0240
40	—	4·0228
30	—	4·0204
20	—	4·0156
19	3·8908	4·0148
13	3·8894	4·0092
10	3·8889	—
0	3·8868	—

The isothermals at 100° c. were completely redetermined twice over by this last procedure to investigate the effect of a longer period of charging. Thus, instead of the two-hour charging period adopted in obtaining the foregoing isothermals, the pressure was maintained at a constant value for four hours in the first case and for eight hours in the second before each exposure.

In all these series of observations the isothermals were found to be of the same form, the only difference between them being that, as the charging period was increased, the range of pressure over which the two phases coexisted also increased, both on raising and on lowering the pressure. For instance, when the charging period was two hours, the two phases existed together, for ascending pressure, between 29·2 and 38 cm. of mercury, and for descending pressure

between 17·2 and 15·0 cm. of mercury, that is, the $(\alpha+\beta)$ region extends over a pressure of 8·8 cm. of mercury with increasing pressure and over only 2·2 cm. of mercury with decreasing pressure. The corresponding figures when the charging period was lengthened to eight hours were 14 cm. and 6 cm. of mercury respectively. The figures recorded in table 1 include mean values of the three sets of observations taken, and thus they show the wider ranges of two-phase regions referred to above.

Within the accuracy of measurement the parameter of the α-phase is found to increase linearly with pressure. It has been shown (Sieverts and Zaff, 1935) that hydrogen dissolves in palladium according to the $\sqrt{p}$ law at very low pressures, so that a departure from a linear relation between pressure and parameter might be expected in this region. The present method is, however, not suitable for investigating, in detail, the relation which holds between these two quantities for very dilute solutions.

(ii) *Hysteresis curves at other temperatures*

It had previously been found (Owen and Jones, 1937 b) that the ascending and descending isothermals at 120° c. were in general agreement with those at 100° c., but that the ranges over which the two phases appeared together, both with ascending and with descending pressures, were wider at 120° c. than at 100° c. under the same conditions of pressure and time in the steady state. This section deals in more detail with this effect at different temperatures.

The isothermals were determined at each of the temperatures 60° c., 80° c., 120° c. and 130° c., but, owing to the experimental arrangement, only a portion of the ascending isothermal could be determined at 130° c. The procedure adopted in each instance was to charge the specimen for two hours before taking the x-ray photograph in accordance with the procedure followed in determining the first set of isothermals at 100° c. described in the previous section.

The isothermals at these temperatures and at 100° c. are shown in figure 2; they are all of the same form. The black symbols represent ascending and the open symbols descending pressures. From a consideration of the curves the following conclusions are reached:—(1) Starting from low pressures, the α-phase parameter increases linearly with increase in pressure of hydrogen. This statement is subject to the limitations referred to at the end of the last section. Owing to the good definition of the lines in the photographs obtained with the α-phase in this region of pressure, the accuracy of parameter measurement is fairly high, namely, ±0·0003 A. The rate of increase of α-parameter with pressure is greater at low than at high temperatures. (2) Up to the pressure where the β-phase appears, the linear relation between α-parameter and pressure holds whether the pressure is increased or decreased. (3) The values of the α-phase parameter and the pressures at which the β-phase appears increase as the temperature increases. (4) If the pressure is lowered after the β-phase has appeared, the α-phase parameter values no longer agree with those obtained before the appearance of the β-phase. (5) The range of pressure over which the $(\alpha+\beta)$ region exists when the pressure is raised increases as the temperature is raised. (6) The parameter of the β-phase lattice when it first appears is approximately the same at all temperatures between 60° c. and 130° c.

(7) With increasing pressure the α-phase disappears when the β-phase parameter attains a value which is approximately the same at all temperatures between 60° c. and 120° c. (8) After the disappearance of the α-phase, the β-phase parameter increases as the pressure is further increased, and ultimately the rate of increase of the β-parameter reaches approximately the same value at all temperatures. (9) When the β-phase is formed at high pressure and the pressure is then decreased, the β-parameter changes with pressure and follows the same curve as that obtained after the disappearance of the α-phase when the pressure was increased, but the pressure has to be lowered beyond the value at which the α-phase disappeared with increasing pressure before this phase appears with decreasing pressure. (10) With decreasing pressure the α-phase appears

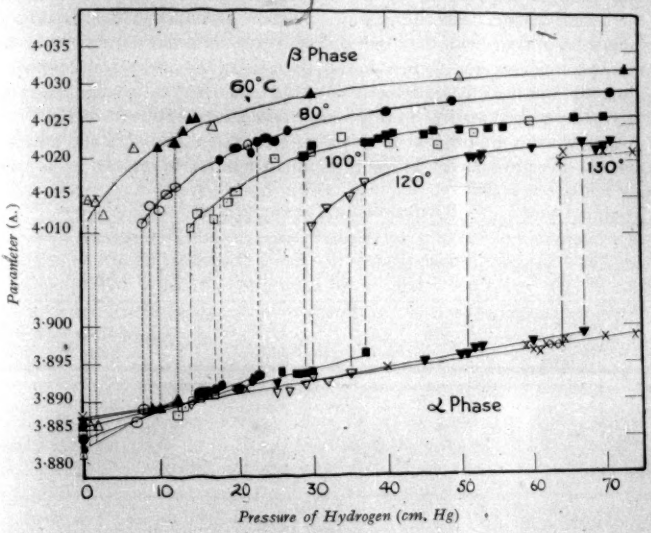

Figure 2.

when the β-parameter has reached a value which is approximately the same for all temperatures between 60° c. and 120° c.; the pressure at which it appears diminishes as the temperature is lowered. (11) As the pressure is further diminished the β-phase rapidly disappears and the α-phase increases in amount. The β-phase entirely disappears after its parameter has reached a definite value, which is practically constant for all the temperatures examined. (12) The α-phase parameter then decreases linearly with decreasing pressure along the curve obtained initially with increasing pressure, but the definition of the lines in the photographs has greatly deteriorated compared with that of the lines obtained with increasing pressure over this range. (13) The range of pressure over which the two-phase region exists when the pressure is decreased narrows as the temperature is lowered. (14) The "hysteresis" loop increases in area

as the temperature is increased. (15) With increasing pressure the parameters of the α- and β-phases increase across the (α+β) region. In an alloy system in a state of equilibrium, the parameters of the components of a two-phase region remain constant within the region. (16) With decreasing pressure the parameters of the α- and β-phases decrease across the (α+β) region. (17) The definition of the lines in the photographs of the β-phase is always poor. At no time did these lines appear well defined.

(iii) *Effect of prolonged charging of the specimens*

From the previous work (Owen and Jones, 1937 b) it was suspected that the relative amounts of the α- and β-phases when they appeared together depended upon the time of charging or of discharging the palladium. For example, after the specimen had been charged at a pressure of 66 cm. for two hours at 100° c. and the pressure was then reduced to 20 cm., and kept at this value for 5, 21 and 72 hours respectively, the values of the parameters of the α- and β-phases were the same, but the relative intensities of the lines from the phases appeared to change. Some information on this point has already been recorded in § 3 (i) of this paper, when isothermals at 100° c. were determined after 2, 4 and 8 hours' charging at a given pressure. It was then found that the hysteresis loops were

Table 2. 17·0 cm. pressure (decreasing) and 100° c.

Film no.	Charging time (hr.)	Parameter (A.) α-phase	β-phase
392	1	$3{\cdot}890_1$	$4{\cdot}012_3$
454	2	$3{\cdot}889_8$	faint
455	5	$3{\cdot}890_3$	,,
393	18	$3{\cdot}889_7$	,,
456	26	$3{\cdot}990_1$	,,
394	42	$3{\cdot}890_9$	,,
395	66	$3{\cdot}889_9$	,,
396	90	$3{\cdot}890_1$	,,

similar in each case, the effect of increasing the charging time being to widen the region of pressure over which the two phases existed together.

It was, however, necessary to make a more comprehensive investigation of the effect on the relative intensity of the α- and β-phase lines, and the apparatus employed for the present work was more suitable than that used previously for this purpose, because now the specimen could be left undisturbed in the chamber for several days and photographs taken at intervals as required. Care was taken that during this period the apparatus was free from leaks and that the temperature was kept at a constant value.

The figures included in table 2 were obtained with a new specimen of palladium charged at 66 cm. pressure and 100° c. for two hours, after which the pressure was reduced to 17·0 cm. and structure spectra were taken after charging the specimen for various periods at this pressure.

It is concluded from these results that for the two-phase region on the

descending isothermal there is no marked change in the relative intensities of the lines after two hours' charging, and the parameter of the α-phase remains constant throughout.

Several other series of observations similar to the above were taken both on descending and ascending isothermals and at different temperatures, the general conclusion being that there is a definite and pronounced change in the relative intensities of the lines from the two phases during the first two hours of charging, but for longer periods, up to 90 hours' exposure, no further measurable change takes place. To carry out these tests, both the pressure and the temperature had to be strictly controlled, as slight variation of both or either, caused changes in the relative intensities of the lines. The more pronounced changes in relative intensities found previously were probably due to the difficulty of keeping the conditions sufficiently steady in the apparatus then used. Other workers, in considering methods of charging the specimen, have arrived at similar conclusions. For instance, Linde and Borelius (1927) claim that in many cases the state of equilibrium was determined not by the method of charging the specimen, but by the subsequent small variations in pressure and temperature. Gillespie and Galstaun (1936) state that it is doubtful whether "there is any theoretical advantage in conducting the additions of hydrogen isothermally. A practical advantage is that of greater consistency of data at a given temperature". One is therefore led to conclude, in view of these statements and the results of the present investigation, that were it possible to maintain the experimental conditions absolutely constant during the prolonged charging of a specimen, there would be no change in parameter value nor, after the first two hours, would there be appreciable change in the relative intensities of the lines from the two phases.

(iv) *Appearance and disappearance of the α- and β-phases*

From the isothermal curves shown in figure 2 the critical data collected in table 3 were obtained. They include the pressures at which the α- and β-phases appeared or disappeared when the pressures were increased or decreased. It should be emphasized that these results were obtained when the pressure of the hydrogen was maintained at a constant value for two hours before exposing the specimen to the x-ray beam. The range of temperature is not as large as would be desired, but over the limited range which was possible with the apparatus employed the data display some interesting relations. For instance, there seems to be a linear relation between the temperature and the logarithm of the "critical" pressure, that is, the "critical" pressure is an exponential function of the temperature.

The parameter of the α-phase—for increasing pressure, when the β-phase appears and when the α-phase disappears, and for decreasing pressures, when the α-phase reappears and when the β-phase disappears—bears a linear relation to the temperature in all cases. The line representing the first appearance of the β-phase and that representing its subsequent disappearance are approximately parallel; also the other two lines relating to the disappearance and reappearance of the α-phase are parallel, but have a different slope from the other lines.

In contrast to the α-phase parameter, the β-phase parameter has approximately a constant value at each phase change; at all the temperatures at which observations were made, its value when it appears on increasing the pressure is about $4{\cdot}020_5$ A. (mean value), and when the α-phase finally disappears about $4{\cdot}022_5$ A. Similar results were obtained with decreasing pressure, although the errors are larger than with increasing pressure. The parameter value of the β-phase is $4{\cdot}013_5$ A. when the α-phase reappears and $4{\cdot}011_6$ A. when the β-phase finally disappears on decreasing the pressure. Thus in each case there is a change of 0·002 A. in the parameter of the β-phase across the two-phase region.

Table 3. Data concerning the appearance and disappearance of the α- and β-phases

	Temp. (° c.)	Parameter (A.) α-phase	Parameter (A.) β-phase	Pressure (cm. Hg)	Log_{10} of pressure
Increasing pressure					
β-phase appears	60	$3{\cdot}889_4$	$4{\cdot}021_6$	10·0	1·000
	80	$3{\cdot}891_8$	$4{\cdot}020_1$	18·0	1·255
	100	$3{\cdot}894_2$	$4{\cdot}020_2$	31·0	1·491
	120	$3{\cdot}896_4$	$4{\cdot}020_4$	51·0	1·707
	130	$3{\cdot}897_3$	$4{\cdot}020_3$	64·0	1·80
α-phase disappears	60	$3{\cdot}890_4$	$4{\cdot}022_9$	12·0	1·079
	80	$3{\cdot}893_6$	$4{\cdot}022_3$	23·0	1·361
	100	$3{\cdot}896_3$	$4{\cdot}022_6$	37·0	1·568
	120	$3{\cdot}899_4$	$4{\cdot}022_2$	68·0	1·832
Decreasing pressure					
α-phase reappears	60	$3{\cdot}885_0$	$4{\cdot}013_9$	2·0	0·301
	80	$3{\cdot}888_5$	$4{\cdot}013_1$	9·0	0·954
	100	$3{\cdot}891_1$	$4{\cdot}013_5$	17·0	1·230
	120	$3{\cdot}893_8$	—	36·0	1·556
β-phase disappears	60	$3{\cdot}884_5$	$4{\cdot}012_2$	1·0	0·000
	80	$3{\cdot}888_1$	$4{\cdot}011_9$	8·0	0·903
	100	$3{\cdot}890_2$	$4{\cdot}011_0$	14·0	1·146
	120	$3{\cdot}892_7$	$4{\cdot}011_4$	30·0	1·477

The range of pressure between the appearance of the β-phase and the disappearance of the α-phase with increasing pressure is greater than that between the reappearance of the α-phase and the disappearance of the β-phase with decreasing pressure at any given temperature. Thus the change of 0·002 A. in the parameter of the β-phase takes place over different ranges of pressure according to the temperature and also according to whether the pressure is increasing or decreasing. It would therefore appear that it is the value of the parameter rather than the pressure that decides when the phase change takes place, and so it is the concentration of hydrogen in palladium that is the deciding factor. Examining the curves in figure 2 from this point of view and

estimating the approximate concentration from the curves of Hoitsema and Roozeboom (1895), this conclusion appears to be confirmed. For example, the concentration (i.e. the ratio of the number of hydrogen to the number of palladium atoms) when the β-phase appears on increasing the pressure ($r_{\alpha\rightarrow}$) is 0·10, 0·10 and 0·095 at 80° C., 100° C. and 120° C. respectively; similarly, the concentrations ($r_{\alpha\leftarrow}$) at these temperatures when the β-phase disappears on decreasing the pressure are $0{\cdot}04_6$, $0{\cdot}04_4$ and $0{\cdot}05_5$. The corresponding figures for $r_{\beta\rightarrow}$ and $r_{\beta\leftarrow}$, although they show the same tendency towards constant values at the different temperatures, are not so reliable, and will not be included here.

The behaviour of palladium charged with hydrogen under various conditions of experiment is, therefore, remarkably definite and regular in the regions where hysteresis occurs, even though the system has not reached its final state of equilibrium.

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A NEW METHOD FOR THE MEASUREMENT OF IMPEDANCE AT ULTRA-HIGH FREQUENCIES USING A SYSTEM OF LECHER WIRES

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MS. received 19 *June* 1943

ABSTRACT. The theory of a new method for the measurement of impedance at ultra-high frequencies is developed and experimental results illustrating its scope are given. The method eliminates many of the disadvantages associated with previous Lecher-wire methods, and appears to be applicable to the direct measurement of the impedance of aerials.

§1. INTRODUCTION

IN the measurement of impedance using a system of Lecher wires, the usual method is to terminate the wires with the unknown impedance and to measure the currents or voltages at various points on the wires. The value of the impedance is then calculated from these observations. The method will give accurate results only if the frequency and power output of the oscillator used in the experiment remain constant during the time observations are being made. To eliminate errors due to fluctuations in the oscillator power output, Flint and Williams (1941) devised a method in which the ratios of the currents flowing in two bridges are used to determine the unknown impedance. One

bridge is fixed at the end of the wires and consists of the unknown impedance in series with a current meter, whilst the other bridge is movable and consists only of a current meter. The method gives good results, but has the disadvantage that the impedance of the current meter connected in the fixed bridge must be known. A subsidiary experiment must, therefore, be performed to determine this impedance.

The following method eliminates errors due to fluctuations in the power output of the oscillator, and also avoids the difficulty discussed above. The unknown impedance, Z, is connected across the end of the wires and the ratios of the currents flowing in two other bridges, Z_1 and Z_2, located at various points on the wires, are used to calculate its value.

Whilst, theoretically, Z_1 and Z_2 can be moved independently, the method becomes practicable only when they are moved together in the form of a tandem bridge with their spacing fixed at a critical value. A subsidiary experiment has to be performed to determine this spacing, although under certain circumstances its value can be calculated. The impedances of Z_1 and Z_2 are not required.

§ 2. THEORY OF THE METHOD

In addition to the notation shown in figure 1, let the impedance of the circuit beyond B not including Z_2 be Z', and that including Z_2 be Z_3, so that

$$Z_3 = \frac{Z_2 Z'}{Z_2 + Z'} \qquad \ldots\ldots(1)$$

The current flowing into the junction at B, a distance s_1 from A, is

$$I_0' = \frac{Z_1 I_1}{Z_3 \cosh Ps_1 + Z_0 \sinh Ps_1} \qquad \ldots\ldots(2)$$

In this expression P denotes the propagation constant of the wires. Attenuation is assumed to be negligible, so that P is an imaginary quantity $j\beta$, where $\beta = 2\pi/\lambda$, λ denoting the wave-length of the oscillations. The quantity Z_0

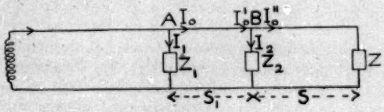

Figure 1.

is the characteristic impedance of the wires. Under the above conditions it is purely resistive, and its value can be calculated from the dimensions of the system.

The ratio of the current entering B to that passing through Z_1 is thus

$$\frac{I_0'}{I_1} = \frac{Z_1}{Z_3 \cosh Ps_1 + Z_0 \sinh Ps_1}.$$

But $$I_0' = I_2 + I_0'' = I_2\left(1 + \frac{Z_2}{Z'}\right).$$

Therefore $$\frac{I_2}{I_1}\left(1 + \frac{Z_2}{Z'}\right) = \frac{Z_1}{Z_3 \cosh Ps_1 + Z_0 \sinh Ps_1}.$$

Replacing Z_3 by the expression given in equation (1), the ratio of the currents in the two bridges Z_1 and Z_2 becomes

$$\frac{I_1}{I_2} = \frac{1}{Z_1}\left(Z_2 \cosh Ps_1 + Z_0 \sinh Ps_1 + \frac{Z_2}{Z'} . Z_0 \sinh Ps_1\right). \quad \ldots\ldots(3)$$

The impedance of the circuit beyond B not including Z_2 is

$$Z' = Z_0 \frac{Z \cosh Ps + Z_0 \sinh Ps}{Z \sinh Ps + Z_0 \cosh Ps}. \quad \ldots\ldots(4)$$

This expression for Z' is simplified by the introduction of the current-reflection coefficient $K = \dfrac{Z_0 - Z}{Z_0 + Z} = e^{-2r}$, when it takes the form

$$Z_0 \tanh (Ps + r).$$

Substituting for Z' in equation (3) gives

$$\frac{I_1}{I_2} = \frac{1}{Z_1}\{Z_2 \cosh Ps_1 + Z_0 \sinh Ps_1 + Z_2 \sinh Ps_1 \coth (Ps + r)\}. \quad \ldots\ldots(5)$$

It is the modulus, ρ, of this ratio which is determined in an experiment. Its mathematical form can be obtained from equation (5) after separating the real and imaginary parts:—

$$\frac{I_1}{I_2} = \frac{1}{Z_1}\{(R_2 + jX_2)\cos \beta s_1 + jZ_0 \sin \beta s_1 + (R_2 + jX_2)j \sin \beta s_1 \coth [a + j(b + \beta s)]\}. \quad \ldots\ldots(6)$$

In this equation R_2 and X_2 are respectively the resistive and reactive series components of Z_2, whilst r, which is in general complex, has been denoted by $a + jb$, so that $Ps + r$ becomes $a + j(b + \beta s)$.

The subsequent calculation of the modulus is simplified by writing the equation in the form

$$\frac{I_1}{I_2} = \frac{1}{Z_1}\{A + jB + (C + jD)\coth [a + j(b + \beta s)]\}. \quad \ldots\ldots(7)$$

where A, B, C and D are quantities which are independent of both the terminating impedance Z and the distance s. They are constants if the bridges Z_1 and Z_2 remain unchanged and if their separation, s_1, is fixed. Their actual values are

$$\left.\begin{aligned} A &= R_2 \cos \beta s_1, & B &= X_2 \cos \beta s_1 + Z_0 \sin \beta s_1, \\ C &= -X_2 \sin \beta s_1, & D &= R_2 \sin \beta s_1. \end{aligned}\right\} \quad \ldots\ldots(8)$$

The resulting expression for the square of the modulus of I_1/I_2 is then

$$\rho^2 = \frac{1}{|Z_1|^2}\left\{(A^2 + B^2) - (C^2 + D^2) + \frac{(C^2 + D^2)\cosh 2a + (AC + BD)\sinh 2a + (AD - BC)\sin 2(b + \beta s)}{\sinh^2 a + \sin^2 (b + \beta s)}\right\}. \quad \ldots\ldots(9)$$

The only variable in this expression is s, so that it can be reduced to

$$\rho^2 = K_1 + \frac{K_2 + K_3 \sin 2(b + \beta s)}{\sinh^2 a + \sin^2 (b + \beta s)}, \quad \ldots\ldots(10)$$

where K_1 and K_3 are constants for a fixed system of the bridges Z_1 and Z_2, and K_2 is a constant involving the impedance Z. The values of K_1 and K_3 can be calculated if the impedances Z_1 and Z_2 are known.

Owing to losses present at high frequencies, each current meter will only measure a fraction of the total current flowing in the bridge of which it is a part. This fraction will be a constant for a given frequency, but will be different for each meter, so that in practice the quantity measured will be $k\rho^2$. If this is substituted for ρ^2 in the above equations, analogous equations are obtained with different values for the constants K_1, K_2 and K_3.

Since, however, the constants will be determined by experiment, equation (10) can remain unchanged; the meter constants need not be the same and their values need not be known.

The expression for ρ^2 given in equation (10) is still rather complex for experimental work, due to the presence of the quantity $K_3 \sin 2(b+\beta s)$ in the numerator of the second term. This quantity, however, will disappear if K_3 can be made zero. This condition is attained if $AD-BC=0$.

On substituting the values for A, B, C and D given in equations (8) the condition becomes

$$R_2^2 \sin \beta s_1 \cos \beta s_1 + X_2^2 \sin \beta s_1 \cos \beta s_1 + Z_0 X_2 \sin^2 \beta s_1 = 0, \quad \ldots\ldots(11)$$

that is

$$[Z_2^2 + Z_0 X_2 \tan \beta s_1] \sin \beta s_1 \cos \beta s_1 = 0,$$

where $Z_2^2 = R_2^2 + X_2^2$.

This equation has two solutions, namely,

$$\sin \beta s_1 = 0 \quad \text{and} \quad \tan \beta s_1 = -Z_2^2/Z_0 X_2.$$

If $\sin \beta s_1$ is made zero, ρ^2 becomes independent of the terminating impedance, so that in practice the condition to be satisfied is

$$\tan \beta s_1 = -Z_2^2/Z_0 X_2. \quad \ldots\ldots(12)$$

The required value of s_1 can thus be calculated if Z_2, Z_0 and X_2 are known. Z_0 can be calculated, whilst Z_2 and X_2, which refer to a current meter, can be determined by the method of Flint and Williams (1941). This calculation is not necessary in practice, since it will be seen later that the correct separation can very easily be obtained experimentally.

When equation (12) is satisfied, the final equation for ρ^2 reduces to

$$\rho^2 = K_1 + \frac{K_2}{\sinh^2 a + \sin^2 (b+\beta s)}. \quad \ldots\ldots(13)$$

This equation can also be written

$$\rho^2 = K_1 + \frac{2K_2}{\cosh 2a - \cos 2(b+\beta s)}. \quad \ldots\ldots(14)$$

In an experiment to determine the values of a and b either equation can be used, the work involved being approximately the same in each case. In the following discussion equation (13) has been adopted. The values of the resistive and reactive components of the unknown impedance are obtained by substituting the values of a and b in the appropriate formulae.

In terms of the notation already introduced,

$$\frac{Z_0-Z}{Z_0+Z}=e^{-2(a+jb)},$$

so that $Z=Z_0\tanh(a+jb)$.

Since Z_0 is purely resistive, the real or resistive component of Z is

$$Z_0\frac{\tanh a\sec^2 b}{1+\tanh^2 a\tan^2 b} \quad \text{or} \quad Z_0\frac{\sinh 2a}{\cosh 2a+\cos 2b} \qquad \ldots\ldots(15)$$

and the imaginary or reactive component is

$$Z_0\frac{\operatorname{sech}^2 a\tan b}{1+\tanh^2 a\tan^2 b} \quad \text{or} \quad Z_0\frac{\sin 2b}{\cosh 2a+\cos 2b} \qquad \ldots\ldots(16)$$

§ 3. DETERMINATION OF THE VALUE OF K_1

Since K_1 is a constant independent of Z, it is advisable, before commencing any experimental investigation, to determine accurately its value. A convenient method is to short-circuit the ends of the Lecher wires so that a and b become zero. Equation (13) then reduces to

$$\rho^2=K_1+K_2\operatorname{cosec}^2\frac{2\pi}{\lambda}s. \qquad \ldots\ldots(17)$$

Thus a graph of ρ^2 plotted against $\operatorname{cosec}^2\frac{2\pi}{\lambda}s$ is a straight line whose intercept on the axis of ρ^2 gives the value of K_1. In practice, Z_1 and Z_2 are chosen to make K_1 as small as possible.

§ 4. DETERMINATION OF THE VALUE OF b

Equation (13) shows that a graph of ρ^2 drawn against s is symmetrical about the turning points provided s_1 is correctly chosen. The values of s at these points can thus be accurately determined. The form of the graph is shown in figure 4.

The maximum values of ρ^2 occur when $\sin^2(b+\beta s)=0$, that is, when $b+\frac{2\pi}{\lambda}s=0,\ \pi,\ \ldots\ldots$, etc., and the minimum values occur when $\sin^2(b+\beta s)=1$, that is, when $b+\frac{2\pi}{\lambda}s=\pi/2,\ 3\pi/2$, etc. The distance between successive maxima and minima is thus $\lambda/4$, so that λ can be obtained accurately from the graph. The value of b can be deduced from the positions of the maxima or minima.

§ 5. DETERMINATION OF THE VALUE OF a

The value of a is best obtained by rewriting equation (13) in the form

$$\frac{K_2}{\rho^2-K_1}=\sinh^2 a+\sin^2(b+\beta s). \qquad \ldots\ldots(18)$$

This shows that if $1/(\rho^2-K_1)$ be plotted against $\sin^2(b+\beta s)$ a linear graph is obtained from which a may be determined. This graph is illustrated in figure 5; the intercept OA gives the value of $\sinh^2 a$.

A more rapid determination of the value of a can be carried out by measuring the maximum and minimum values of ρ^2. These are $\rho^2_{max.} = K_1 + \frac{K_2}{\sinh^2 a}$ and $\rho^2_{min.} = K_1 + \frac{K_2}{\cosh^2 a}$, respectively.

Thus
$$\frac{\rho^2_{min.} - K_1}{\rho^2_{max.} - K_1} = \tanh^2 a, \qquad \ldots\ldots(19)$$

from which a can be calculated.

§ 6. ADJUSTMENT OF THE BRIDGES Z_1 AND Z_2

The separation of the bridges Z_1 and Z_2, necessary to make K_3 zero, can be calculated from equation (12). It is more convenient, however, to obtain the correct separation experimentally by a method in which the values of Z_1 and Z_2 are not required.

If the ends of the Lecher wires are "shorted" and K_3 is not zero, equation (10) becomes
$$\rho^2 = K_1 + K_2 \operatorname{cosec}^2 \beta s + 2K_3 \cot \beta s. \qquad \ldots\ldots(20)$$

Under these conditions a graph of ρ^2 plotted against s is not symmetrical about the turning points, but is of the form shown in figure 2. Curve A is obtained if the separation makes K_3 positive and curve B if it makes K_3 negative.

The curve is symmetrical only if K_3 is zero, in which case the minimum value of ρ^2 occurs when $s = \lambda/4$.

Therefore, before determining K_1, three values of ρ^2 are obtained, one for a value of s less than $\lambda/4$, say $(\lambda/4 - 5)$ cm.; another when $s = \lambda/4$ cm.; and a third

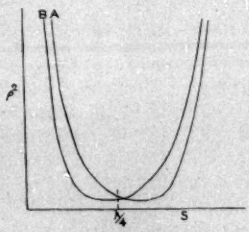

Figure 2.

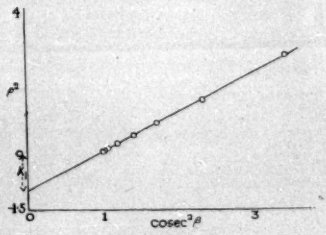

Figure 3.

when $s = (\lambda/4 + 5)$ cm.; and the separation of the bridges is adjusted until the value of ρ^2 at $s = \lambda/4$ cm. is a minimum. The final adjustment can be made by obtaining the values of ρ^2 in more positions.

It is found that if Z_1 and Z_2 are suitably chosen, the separation of the bridges is not critical and that the results obtained for an impedance are not affected appreciably by a small error in this adjustment.

§ 7. APPLICATION OF THE METHOD

The method has been used to measure the impedances of carbon resistors and to investigate the properties of liquids, especially the values of their dielectric constants, at frequencies of the order of 150 megacycles per sec. It has also

been used to measure the capacities of small condensers such as are formed by the electrodes of a valve.

The Lecher wires were loosely coupled to a line-controlled oscillator. Each movable bridge consisted of a vacuum thermojunction, the output terminals of which were connected to a micro-ammeter. Thus the ratio of the meter readings gave directly the value of $k\rho^2$. Referring to figure 1, thermojunction Z_1 had a d.c. resistance of 410 ohms, and thermojunction Z_2 a d.c. resistance of 1530 ohms. The Lecher wires had a characteristic resistance of 328 ohms, attenuation being negligible at the above frequency.

A preliminary experiment was performed to determine the correct separation of the thermojunctions at the wave-length used, which was 220 cm. The wires were "shorted" with a polished copper plate soldered to their ends, and adjustments were carried out as described in the preceding part of the paper. The separation necessary to make K_3 zero was found to be 45 cm.

§ 8. EXPERIMENTAL DETERMINATION OF THE VALUE OF K_1

With the ends of the Lecher wires still "shorted", a series of values of ρ^2 was obtained for values of s varying from 10 cm. to 200 cm. The value of the maxima of ρ^2 approached infinity, showing that the resistive component of the shorting plate was negligible, whilst the positions of the maxima and minima showed that the reactive component was also negligible. The graph of ρ^2 plotted against $\operatorname{cosec}^2 \frac{2\pi}{\lambda} s$ is shown in figure 3. Each value of ρ^2 plotted is the mean of a number of values obtained for the same value of $\operatorname{cosec}^2 \frac{2\pi}{\lambda} s$. The value of K_1 obtained from the graph is -0.95.

§ 9. MEASUREMENT OF THE IMPEDANCE OF A CARBON RESISTOR

In the case to be described, the carbon resistor had a d.c. resistance of 500 ohms. It was used to terminate the Lecher wires, and a series of values of ρ^2 was obtained for different values of s. The graph of ρ^2 against s is shown in figure 4.

The readings of s at the maxima are 42 cm. and 152 cm. and at the minimum 97 cm., giving the value 220 cm. for the wave-length. The value of b deduced from these positions is consequently $-68°{\cdot}7$.

The values of $\sin^2\left(b+\frac{2\pi}{\lambda}s\right)$ can now be calculated. The graph obtained when $1/(\rho^2-K_1)$ is plotted against this quantity is shown in figure 5. Some of the points are too close together to be distinguished on the graph. The specimen points shown are those which occur between the first maximum and the minimum of the curve of figure 4.

The numerical value of the intercept on the $\sin^2\left(b+\frac{2\pi}{\lambda}s\right)$ axis is 0·715, which is the value of $\sinh^2 a$.

The value of the resistive component of the resistor is calculated from

expression (15), and the value of the series reactive component from expression (16). The values are given below:

Resistive component = 429 ohms;
Series reactive component = −131 ohms.

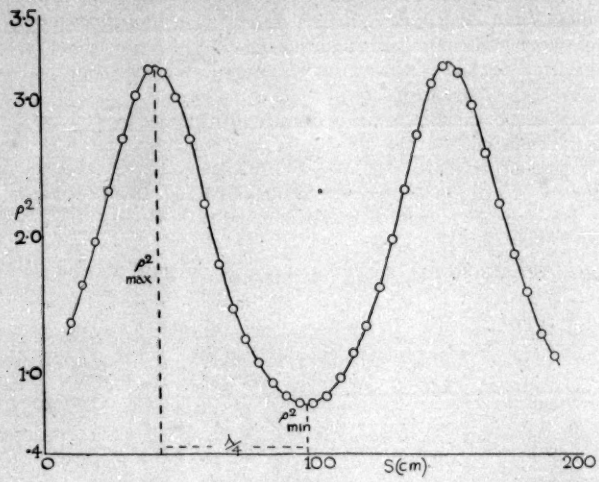

Figure 4.

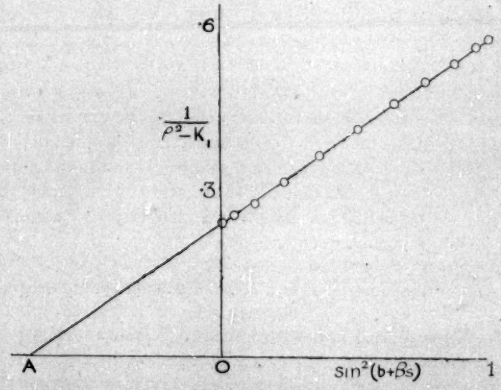

Figure 5.

The reactive component is thus capacitative. The value of the parallel self-capacity of the resistor can be calculated from the value of the equivalent series reactance given above.

A much more rapid method of determining *a* is to use equation (19). The

values then obtained are 430 ohms for the resistive component and − 134 ohms for the reactive component. These values show that this method is capable of giving good results.

The method provides a rapid means of determining the nature of the reactance of an impedance. If the reactance is capacitative, the first turning point is a maximum, whilst if it is inductive, the first turning point is a minimum. If the first maximum occurs when s is zero, the impedance is purely resistive.

§ 10. INVESTIGATION OF THE PROPERTIES OF A LIQUID

The following results were obtained for the properties of a transformer oil which, at a frequency of 500 cycles per sec., has a dielectric constant of 2·06. The wires were first terminated with an air condenser having a capacity of between 2 and 3 $\mu\mu$F., and a series of values of ρ^2 was obtained for various values of s. The condenser was then immersed in the transformer oil and a further

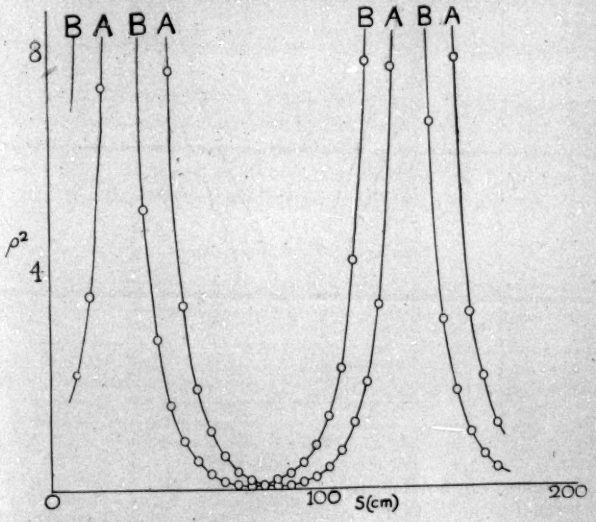

Figure 6.

series of values of ρ^2 was obtained. The curves of ρ^2 plotted against s are shown in figure 6. Curve A was obtained with air as the dielectric and curve B with the oil as the dielectric of the terminating condenser. The values of ρ^2 near the maxima are large and have not been plotted on the graph.

The values of the maxima of ρ^2 approach infinity, showing that the losses in the oil are very small; a is almost zero.

The values of b can be obtained from the positions of the maxima or minimum.

Somewhat greater precision in the estimation of b can be obtained by plotting $\log(\rho^2)$ against s. The minimum is then given by a discontinuity, figure 7.

The values of b obtained from these graphs are: $b_1 = -54° 0'$ with air as the dielectric, and $b_2 = -36° 0'$ with oil as the dielectric, the wave-length being still 220 cm.

When a can be assumed to be zero, as in the present case, the expression for the series resistive component of the terminating impedance becomes zero and that for the reactive component becomes simply $Z_0 \tan b$. The formula for the dielectric constant is then

$$\epsilon = \tan b_1 / \tan b_2. \qquad \ldots\ldots(21)$$

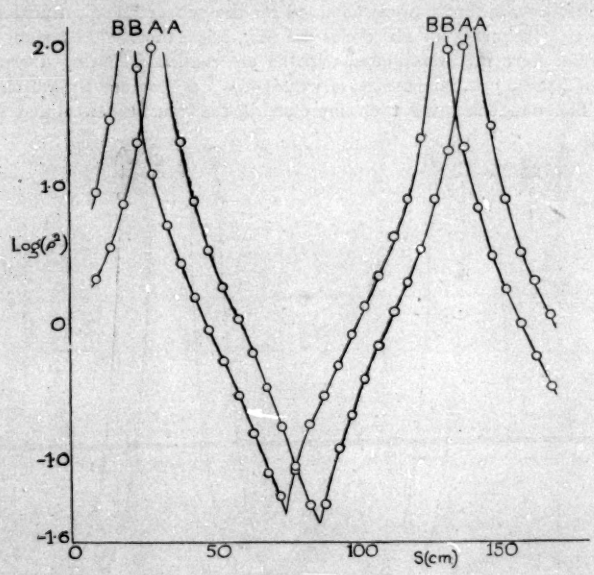

Figure 7.

Thus the value of Z_0 is not required.

The value of the dielectric constant of the transformer oil calculated from the above results is 1·89.

In an experiment all that is required is the value of s at the first maximum of ρ^2, first with air and then with the liquid as the dielectric of the terminating condenser. These values can be determined quickly and accurately.

ACKNOWLEDGMENTS

In conclusion I wish to thank Dr. H. T. Flint and Mr. D. Rogers for their interest in the method.

REFERENCE

FLINT and WILLIAMS, 1941. *Phil. Mag.* 32, 489.